Fourier Transforms and their Application to Pulse Amplitude Modulated Signals

Phil Lucht
Rimrock Digital Technology, Salt Lake City, Utah 84103
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rimrock@xmission.com

Maple code is available upon request. Comments and errata are welcome.
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The graphics look ratty in Windows Adobe PDF viewers when not scaled up, but look just fine in this excellent freeware viewer: http://www.tracker-software.com/pdf-xchange-products-comparison-chart .

The table of contents has live links.

Overview and Summary

The Fourier Integral Transform and its various brethren play a major role in the scientific world. This monograph develops the analog and digital theory of these transforms and applies that theory to pulse-amplitude-modulated (PAM) signals referred to as "pulse trains" -- signals formed from a single arbitrary pulse shape, $x(t) = \Sigma_n y_n x_{pulse}(t-nT_1)$. Particular attention is paid to the spectral power density $\mathscr{P}(\omega)$ of pulse trains which form a statistical ensemble. When PAM signals are passed through a "linear time-invariant" circuit or other apparatus, that apparatus may be viewed as a "filter" and, due to the Convolution Theorem, the behavior of such a filter is most easily understood in the frequency domain.

All calculations are done in line for the reader to see and perhaps critique. This is done to provide a clear tracing path for the repair of errors, to demonstrate unusual techniques, and hopefully to remove some of the mystery associated with Fourier Transform mathematics. With just a few exceptions, every equation appearing in this document is derived in this document.

The reader is assumed to be familiar with calculus and complex integration.

Equations which are quotes of earlier equations have their equation numbers in italics.

A detailed summary of the material presented appears at the end of this document. Here we provide only a brief chapter-level summary.

Chapter 1 (Sec 1-13) develops the basic theory of the Fourier Integral Transform and its Sine and Cosine cousins. This Chapter forms the underpinning of all subsequent Chapters. The Convolution Theorem receives special attention. The connection is made between the Laplace Transform and the "generalized" Fourier Transform applied to causal functions. Various "rules" are derived, and a connection is made between filter spectra and time-domain Green's Functions.

Chapter 2 (Sec 14-19) examines the Fourier Integral Transform spectrum of a simple pulse train formed from a general pulse shape. Consideration of pulse trains of infinite length leads to a derivation of the Fourier Series Transform. The chapter concludes with a discussion of sample pulse trains formed from box and bi-phase pulses.

Chapter 3 (Sec 20-27) deals with various digital forms of the Fourier Transform and their corresponding convolution theorems and applies these concepts to amplitude-modulated pulse trains (PAM signals) and to digital filters. Topics include image spectra, aliasing and Nyquist rate, group delay, FIR and IIR filters, poles, and impulse response. The first digital transform is called the Digital Fourier Transform which is an ω-domain version of the Z Transform whose variable is $z = e^{i\omega\Delta t}$. The Z Transform and Discrete Fourier Transforms are then addressed for both periodic and aperiodic signals. A recurring example is a simple RC filter section.

Chapter 4 (Sec 28-30) shows that symmetric FIR filters have linear phase and thus constant group delay. A specific brick wall digital filter is designed and then later used as an oversampling interpolation filter in the design of a D/A converter output section. This system is then simulated with a simple Maple program.

Chapter 5 (Sec 31) explores the subject of dispersion relations for the spectral function X(ω) and for other related functions treated as analytic functions of a complex variable.

Chapter 6 (Sec 32-38) derives expressions for the energy and power, and the spectral energy and power densities of an amplitude-modulated pulse train. This work is carried out with a moderate amount of mathematical rigor. Correlation and autocorrelation are mentioned. The notion of a statistical pulse train is presented and the spectral power density of such pulse trains is established. These results are then applied to various uncorrelated standard line codes including NRZ, RZ and Manchester. Two examples of correlated pulse trains are then treated -- AMI and Change/Hold -- and the latter is then used to obtain results for the NRZI line code.

Appendix A discusses delta functions at a someone deeper and more practical level than is commonly found in texts and on the web. Delta function models are constructed and many mathematical identities are developed which find use in the main text.

Appendix B derives an obscure identity used in the Discrete Fourier Transform discussion of Section 27.

Appendix C further develops Fourier Integral Transform theory beyond the treatment of the main text. Instead of using the notation f(t) and F(ω), here we use f(t) and f^(ω).

Appendix D computes a certain sum needed in Section 35.

Appendix E provides a table of all transform pairs appearing in this document and shows how they are related to each other.

Appendix F explores the properties of infinite pulse trains which consist of a repeated subsequence of P elements. A square wave and an MLS sequence are examples.

Appendix G treats random variables and provides a brief review of probability theory. Experiments considered include dice, a spinner, babies who grow up, and finally the experiment of an Apparatus that generates sequences used as pulse train amplitudes.

Chapter 1: The Fourier Integral Transform and Related Topics

The purpose of this chapter is to demonstrate the use of certain mathematical tools associated with the Fourier Integral transform. Along the way simple examples are considered, with emphasis on the particular example of an isolated square pulse in the time domain. If we can make things work out for a square pulse, we can presumably go on to harder problems with the same tools.

One normally analyzes a square-wave pulse train using a Fourier Series, since such a pulse train is a periodic function. In Chapter 2 below, we will make the connection between the conventional Fourier Series, and our Fourier Integral approach.

1. The Fourier Integral and Sine/Cosine Transforms

(a) Pulses and Pulse Trains, Periodic and Aperiodic

Before starting, we need to define a few basic terms describing a function x(t) :

x(t) is a "pulse" if x(t) decays to zero at both $t = \pm\infty$. Normally we think of a pulse as having a finite extent, but it could be something like a Gaussian pulse with an infinite extent. Such pulses fall into the class of aperiodic (non-periodic) functions. Further restrictions on x(t) will be given below.

x(t) is a "simple pulse train" if it is constructed as a sum of identical pulses each of which is shifted by the same constant amount T_1 from the previous pulse. Simple pulse trains which are infinite in extent then fall into the class of periodic functions. If finite in extent, they are aperiodic.

x(t) is a "general pulse train" if the pulses are allowed to have arbitrarily different amplitudes. We refer to this as an amplitude-modulated pulse train. If the pulse train is infinite and the amplitude-modulation is a repeating pattern (such as in a square wave), the pulse train is periodic. If the amplitudes are random or are, say, the decimals of π, the pulse train is aperiodic. We do not treat pulse trains composed of pulses having different shapes, such as would be encountered in frequency or phase shift keying, although the methods presented can be modified to account for such pulse trains.

(b) The Fourier Integral Transform X(ω)

Strictly speaking, the Fourier Integral Transform only applies to aperiodic functions due to the integrability condition given below, but if we ignore that condition and blindly apply the transform to a periodic function, the limit of the Fourier Integral Transform becomes the Fourier Series Transform, as will be demonstrated below.

We now state the Fourier Integral transform. Let x(t) be some function of time. If we define X(ω) to be the "spectral components" or "spectrum" of x(t) according to (1.1), then the claim is that we can recover x(t) from these spectral components according to (1.2): [conventions are discussed in Section 5]

Fourier Integral Transform:

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

Dimensions: If Dim[x(t)] = V, then Dim[X(ω)] = V-sec. Here "V" could by anything or nothing. Often we shall regard x(t) as dimensionless, in which case V is "nothing", but V does suggest Volts as a typical unit for x(t) one might encounter in practice.

In equivalent language, (1.2) represents an expansion of x(t) in terms of the spectral components X(ω). Equation (1.1) shows how these components are "projected out" of the function x(t). Sometimes this projection (1.1) is called "the transform" and then (1.2) is "the inverse transform" or "inversion formula" or "recovery formula" in the sense that x(t) is recovered from its spectral components. The variables t and ω are referred to as "conjugate variables". In this document, we shall think of t as time and ω as angular frequency, but they could be arbitrary conjugate variables. In the theory of waves, they might be position x and wavenumber k.

The expansion (1.2) can be rewritten in terms of frequency f = ω/2π (so df = dω/2π) as follows:

$$\mathcal{X}(f) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i2\pi ft} \qquad \text{projection = transform} \qquad (1.3)$$

$$x(t) = \int_{-\infty}^{\infty} df\, \mathcal{X}(f)\, e^{+i2\pi ft} \qquad \text{expansion = inverse transform} \qquad (1.4)$$

where $\mathcal{X}(f) = X(\omega) = X(2\pi f)$. This form gets rid of the (1/2π) in (1.2), but sticks us with 2π factors in the exponents. In general we shall stick with the ω form.

There are restrictions on the function x(t) (or equivalently, on X(ω) going in the other direction). One restriction is that x(t) must be "piecewise continuous", which allows x(t) to have isolated places where it is discontinuous such as at the edges of our box pulse considered below. A second restriction is that the *derivative* of x(t) must also be piecewise continuous. If t is a discontinuous point (such as an edge of our box), one must interpret x(t) in (1.2) as $\lim_{\varepsilon\to 0}$ [x(t+ε) + x(t-ε)]/2. This is why one often sees the Heaviside step function θ(t) with the property θ(0) = 1/2, as will be demonstrated later.

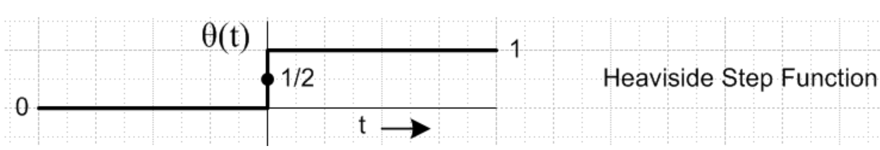


Fig 1.1

The Heaviside step function is often denoted by u(t) or H(t), but we shall always use θ(t).

A third condition is that x(t) must be "L_1 integrable" which means this:

$$\int_{-\infty}^{\infty} dt\, |x(t)| < \infty \qquad \text{// that is, this integral must be finite} \qquad (1.5)$$

Notice that x(t) = sin(t) is *not* L_1 integrable, although x(t) = sin(t) $e^{-\varepsilon|t|}$ is for any tiny ε > 0. Certainly any finite amplitude pulse of any shape having a finite temporal extent will be L_1 integrable. If we consider the Fourier Transform of a periodic function in the ε limit sense just stated, then we may apply the Fourier Transform to periodic functions as well as aperiodic ones. This ε limit sense is directly associated with the theory of distributions, and that is why lots of delta functions appear in the analysis.

Appendix C provides more detail on the Fourier Integral Transform. It seemed best to keep this material out of the main document flow, though it is quite important. See also Stakgold Vol. 2 Section 5.6.

There are various names associated with this subject, including Fourier, Riemann, Lebesgue, Fubini, Parseval and Plancherel. A special class of functions for which Fourier Transforms are guaranteed to work are the Schwartz Functions. It is a story that goes on and on. For example, the Uncertainty Principle of quantum mechanics is directly associated with the Fourier Integral Transform where conjugate variables are x and p (position and momentum) or t and E= ℏω (time and energy). If one tries to localize x(t), X(ω) spreads out and vice versa. Force light to go through a pinhole and this positional confinement causes uncertainty in photon momentum and the light beam diffracts out from its original center line path through the hole.

In what follows, we shall often interchange the order of two integrations in an expression, or the order of two sums, or of one sum and one integral. When functions are reasonable and integration or summation endpoints are finite, this is always an allowed procedure. When endpoints are infinite, there is some danger that the interchange gives wrong results. Essentially, a sum or integral with an infinite endpoint (or endpoints) is a limiting process, and one is then talking about interchanging the order of two limits. This is a rather technical subject having to do with so-called uniform convergence. A certain Moore-Osgood Theorem says that interchange is allowed as long as both limits exist and at least one of the limits is uniformly convergent. The situation is further complicated by what we above called the "ε limit sense" of distribution theory which in effect makes slightly non-convergent forms be convergent. Suffice it to say that all our order interchanges are justified providing the integrands like x(t) respect the conditions stated above.

(c) The Fourier Sine and Cosine Transforms $X_S(\omega)$ and $X_C(\omega)$

Any function x(t) can be decomposed into even and odd parts under t ↔ -t,

$$x(t) = x_{even}(t) + x_{odd}(t) = [x(t) + x(-t)]/2 + [x(t) - x(-t)]/2 \quad . \qquad (1.6)$$

For $x_{even}(t)$, only the cos(ωt) part of (1.1) contributes, and for $x_{odd}(t)$ only the sin(ωt) part. Thus,

$$X_{even}(\omega) = 2 \int_0^{\infty} dt\, x_{even}(t) \cos(\omega t)$$

$$X_{odd}(\omega) = 2 \int_0^{\infty} dt\, x_{odd}(t) \sin(\omega t)$$

where the factor of 2 arises from reflecting the negative part of the integral to the positive side.

Clearly $X_{even}(\omega)$ is even in ω, and $X_{odd}(\omega)$ is odd in ω. Therefore, the *inverse* transformations can be restated in terms of cos and sin in this same manner, where now (1/2π) 2 = (1/π),

$$x_{even}(t) = (1/\pi) \int_0^\infty d\omega\, X_{even}(\omega) \cos(\omega t)$$

$$x_{odd}(t) = (1/\pi) \int_0^\infty d\omega\, X_{odd}(\omega) \sin(\omega t) \quad .$$

Another view to take of these transforms is to regard x(t) as an arbitrary starting function which is defined only for t ≥ 0. One can then *by fiat* add a left side to the function, thereby making it either even or odd as desired. For example, if x(t) = exp(-t) for t > 0, one could either "evenize" or "oddize" the function in this manner

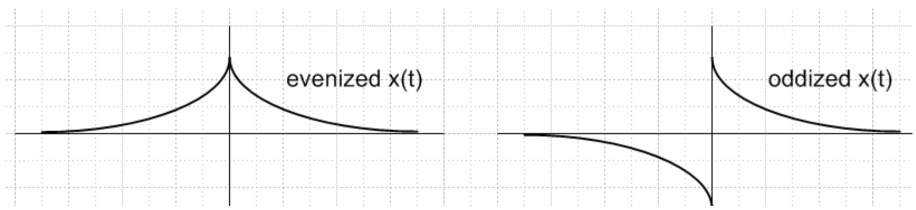


Fig 1.2

Since the projections shown above only make use of data for t ≥ 0, this process is just something the user does mentally to explain why the following two transforms are valid for an arbitrary x(t) which is defined only for t ≥ 0. The inverse transforms if examined at t < 0 will produce left sides for x(t) having the appropriate symmetry as suggested in the above figure.

$$X_c(\omega) = 2 \int_0^\infty dt\, x(t) \cos(\omega t) \qquad \text{Fourier Cosine Transform}$$

$$x(t) = (1/\pi) \int_0^\infty d\omega\, X_c(\omega) \cos(\omega t) \qquad (1.7)$$

$$X_s(\omega) = 2 \int_0^\infty dt\, x(t) \sin(\omega t) \qquad \text{Fourier Sine Transform}$$

$$x(t) = (1/\pi) \int_0^\infty d\omega\, X_s(\omega) \sin(\omega t) \qquad (1.8)$$

One can add an arbitrary factor A to the projection and 1/A to the inversion which will rescale the multiplicative constants, but the product of the constants must be (2/π).

As noted, any x(t) defined over *all* t (-∞,∞) can be decomposed into its even and odd parts, and then one can use tables of Fourier Sine and Cosine Transforms to compute the *complete* Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, [\, x_{even}(t) + x_{odd}(t)\,]\, [\cos(\omega t) - i \sin(\omega t)]$$

$$= \int_{-\infty}^{\infty} dt\, x_{even}(t) \cos(\omega t) - i \int_{-\infty}^{\infty} dt\, x_{odd}(t) \sin(\omega t)$$

$$= 2 \int_{0}^{\infty} dt\, x_{even}(t) \cos(\omega t) - 2i \int_{0}^{\infty} dt\, x_{odd}(t) \sin(\omega t)$$

$$= X_{even,c}(\omega) - i\, X_{odd,s}(\omega) . \qquad (1.9)$$

Extensive tables of Fourier Sine and Cosine transforms appear in Erdélyi Vol. 4 (ET I).

<u>Example</u>: $x(t) = e^{-at}$ with $\mathrm{Re}(a) > 0$:

$$X_c(\omega) = 2 \int_{0}^{\infty} dt\, x(t) \cos(\omega t) = 2 \int_{0}^{\infty} e^{-at} \cos(\omega t) = 2a/(\omega^2+a^2) \qquad \text{FCT}$$

$$x(t) = (1/\pi) \int_{0}^{\infty} d\omega\, X_c(\omega) \cos(\omega t) = (2a/\pi) \int_{0}^{\infty} d\omega \cos(\omega t) /(\omega^2+a^2) = e^{-at}$$

$$X_s(\omega) = 2 \int_{0}^{\infty} dt\, x(t) \sin(\omega t) = 2 \int_{0}^{\infty} e^{-at} \sin(\omega t) = 2\omega/(\omega^2+a^2) \qquad \text{FST}$$

$$x(t) = (1/\pi) \int_{0}^{\infty} d\omega\, X_c(\omega) \sin(\omega t) = (2/\pi) \int_{0}^{\infty} d\omega \sin(\omega t)\, \omega /(\omega^2+a^2) = e^{-at}$$

$$X(\omega) = X_{even,c}(\omega) - i\, X_{odd,s}(\omega) = 2a/(\omega^2+a^2) - i\, 2\omega/(\omega^2+a^2) = 2/(a+i\omega) .$$

Below we shall discuss how the Fourier Integral Transform becomes the Fourier Series Transform for periodic functions. In the same manner, the Fourier Sine Transform becomes the Fourier Sine Series Transform, and similarly for the Cosine transform, though we shall not explicitly discuss these cases.

2. Proof of the Fourier Integral Transform

A simple proof of the Fourier Integral theorem follows from this fact,

$$\int_{-\infty}^{\infty} dx\ e^{\pm ikx} = 2\pi\delta(k) \tag{2.1}$$

where δ(k) is a "distribution" or "symbolic function" known as the Dirac delta function. To "prove" (2.1), we first note that when k = 0, both sides are infinite, which seems promising. When k ≠ 0, the usual arm-waving argument is that the oscillating phasor integrates to 0 over the long haul, or perhaps one claims that instead for the cos(kx) + isin(kx) real and imaginary parts. The "ε limit sense" mentioned above strengthens the arm-waving argument. For example, for k ≠ 0,

$$\text{limit}_{\varepsilon\to 0}\ [\int_0^{\infty} \cos(kx)\ e^{-\varepsilon x}\ dx\ ] \ = \ \text{limit}_{\varepsilon\to 0}\ [\varepsilon / (\varepsilon^2+k^2)\ ]\ = 0\ . \qquad k \neq 0$$

To establish the 2π in (2.1), integrate both sides from k = -a to k = a. The right side gives 2π since the area "under" δ(k) = 1. The LHS gives (here is our first order interchange),

$$\int_{-a}^{a} dk \int_{-\infty}^{\infty} dx\ e^{\pm ikx} = \int_{-\infty}^{\infty} dx \int_{-a}^{a} dk\ e^{\pm ikx} = \int_{-\infty}^{\infty} dx\ [\int_{-a}^{a} dk\ \cos(kx)] \qquad // \sin(kx) \text{ is odd}$$

$$= \int_{-\infty}^{\infty} dx\ [\ 2\sin(ax)/x\ ]\ = 2\ [\int_{-\infty}^{\infty} dx\ \sin(ax)/x\ ]\ = 2\ [\ \pi\ ] = 2\pi\ .$$

More serious derivations of (2.1) are presented in Appendix A (a) which the reader is encouraged to peruse. This Appendix also discusses the meaning of δ(0), a symbol we shall be using in Chapter 6.

One key property of the delta function is its "sifting property",

$$\int_a^b dx\ \delta(x-y)f(x) = f(y)\theta(b-y)\theta(y-a) = f(y)\Theta(a\le y\le b) \qquad a < b \tag{2.2}$$

where θ(x) is the Heaviside Step Function noted above, and $\Theta(a\le y\le b) \equiv \theta(b-y)\theta(y-a)$ is a special notation explained in Appendix A (e) which makes certain manipulations easier to visualize. These functions cause the integral to vanish if y lies outside the range (a,b). If y coincides with endpoint b, say, then since θ(0) = 1/2, the right side becomes f(y)/2, as if the integral were picking up half the area of the delta function. A special case of the above equation is

$$\int_{-\infty}^{\infty} dx\ \delta(x-y)f(x) = f(y)\ . \tag{2.3}$$

Accepting (2.1), we can verify the Fourier Integral Transform in both directions. First (and here are more order interchanges!),

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, \{(1/2\pi) \int_{-\infty}^{\infty} d\omega'\, X(\omega')\, e^{+i\omega' t}\}\, e^{-i\omega t}$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, X(\omega')[\int_{-\infty}^{\infty} dt\, e^{+i(\omega'-\omega)t}] = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, X(\omega')\, 2\pi\, \delta(\omega'-\omega)$$

$$= \int_{-\infty}^{\infty} d\omega'\, X(\omega')\, \delta(\omega'-\omega) = X(\omega)\ .$$

Going the other way is similar,

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, \{ \int_{-\infty}^{\infty} dt'\, x(t')\, e^{-i\omega t'} \}\, e^{+i\omega t}$$

$$= (1/2\pi) \int_{-\infty}^{\infty} dt'\, x(t')\, [\int_{-\infty}^{\infty} d\omega\, e^{+i\omega(t-t')}] = (1/2\pi) \int_{-\infty}^{\infty} dt'\, x(t')\, 2\pi\, \delta(t-t')$$

$$= \int_{-\infty}^{\infty} dt'\, x(t')\, \delta(t-t') = x(t)\ .$$

Have we "proved" the Fourier Transform by doing these verifications? Yes, but we have not proven in detail that the restrictions stated above on x(t) must be respected. In general, "proving" the viability of a transform lies in the realm of Sturm-Liouville theory, see following Comments. The main idea is that one must show that a set of basis functions is "complete" for an interval of interest, which means one must know the full "spectrum" of a certain operator L.

Comments

The set of functions $e^{ikx}/\sqrt{2\pi}$ form a complete orthonormal set on the interval $(-\infty,\infty)$ for functions f(x) of the restricted class described above (L_1 integrable, etc). Picking one of the signs, we can write (2.1) in these two ways, where * means complex conjugation (perhaps think of x as t, and k as ω)

$$\int_{-\infty}^{\infty} dx\, [e^{ikx}/\sqrt{2\pi}\,]\, [e^{ik'x}/\sqrt{2\pi}\,]^* = \delta(k-k') \qquad \text{// functions } e^{ikx}/\sqrt{2\pi} \text{ are orthonormal}$$

$$\int_{-\infty}^{\infty} dk\, [e^{ikx}/\sqrt{2\pi}\,]\, [e^{ikx'}/\sqrt{2\pi}\,]^* = \delta(x-x') \qquad \text{// functions } e^{ikx}/\sqrt{2\pi} \text{ are complete}$$

In general, every "self-adjoint" linear differential operator L on a given interval (a,b) defines a complete orthonormal set of functions on that interval *and* an associated transform on that interval. These functions are the normalized eigenfunctions of the eigenvalue equation $Lu_\lambda = \lambda u_\lambda$. The combination of L and (a,b) is said to define a "Sturm-Liouville problem". When the endpoints are finite and L is "regular" at these endpoints, the spectrum for λ is discrete. When the endpoints are "singular", as for example when one or both are infinite, the spectrum for λ is usually all continuous, but sometimes there is also a discrete component.

In the case of the Fourier Transform, which is our only transform family of interest, $L = -d^2/dx^2$,

$\lambda = k^2$, the interval is $(-\infty,\infty)$, the eigenvalue equation is $-d^2u_k/dx^2 = k^2u_k$ and $u_k = e^{ikx}/\sqrt{2\pi}$.

There are many other "name-brand" transforms and each is associated with a particular L on a particular interval. An example is the Legendre Polynomial Transform on the interval (-1,1). Another is the Fourier Series Transform on some (a,b). Just as we expand x(t) on the $e^{-i\omega t}$ in (1.2) for interval $(-\infty,\infty)$, so also can we expand f(z) on $P_\ell(z)$ for z in (-1,1), or f(x) on $\sin(n\pi x/L)$ for x in (0,L). In the latter two cases, the spectrum is indicated by ℓ = 1,2,3... or n = 1,2,3.. , while in the former ω = real, a continuous spectrum.

The subject has further extension to functions of more than one variable. For example, a function $f(\theta,\varphi)$ where θ,φ define points on a sphere may be expanded on the "spherical harmonics" $Y_{\ell m}(\theta,\varphi)$ which are simultaneously eigenfunctions of two self-adjoint differential operators called L^2 and L_z (angular momentum). The interval for θ is $(0,\pi)$ and for φ is $(0,2\pi)$.

Stakgold Volume I discusses the spectra of differential operators in Chapter 4, and the theory of distributions (such as the delta function) in Chapter 1. Volume II then extends these ideas to multiple variables.

These Comments are only for the reader's possible interest and are not "used" anywhere below except where it is noted that the Fourier Transform functions $e^{-i\omega t}$ form a complete set on the interval $(-\infty.\infty)$.

3. The Convolution Theorem and its Derivation

Suppose three functions of t are related as follows (a convolution integral):

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \text{sometimes written} \qquad a = b * c \tag{3.1}$$

Letting t" = t - t' this can also be written

$$a(t) = \int_{-\infty}^{\infty} dt''\, b(t'')c(t-t'') = \int_{-\infty}^{\infty} dt'\, b(t')c(t-t') = \int_{-\infty}^{\infty} dt'\, c(t-t')\, b(t') \tag{3.2}$$

which just shows that the integral is invariant under the change b ↔ c (so a = b * c = c * b).

Now assume that Fourier Integral expansions exist for a(t), b(t) and c(t) so we can write,

$$a(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{+i\omega t} \qquad A(\omega) = \int_{-\infty}^{\infty} dt\, a(t)\, e^{-i\omega t}$$

$$b(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega t} \qquad B(\omega) = \int_{-\infty}^{\infty} dt\, b(t)\, e^{-i\omega t}$$

$$c(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, C(\omega)\, e^{+i\omega t} \qquad C(\omega) = \int_{-\infty}^{\infty} dt\, c(t)\, e^{-i\omega t} \tag{3.3}$$

Then apply the operation $\int_{-\infty}^{\infty} dt\, e^{-i\omega t}$ to both sides of (3.1),

$$\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, a(t) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left[\int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \right]$$

or

$$A(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left[\int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \right] . \tag{3.4}$$

Next, these expressions follow from (3.3),

$$b(t-t') = (1/2\pi) \int_{-\infty}^{\infty} d\omega''\, B(\omega'')\, e^{+i\omega''(t-t')}$$

$$c(t') = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, e^{+i\omega' t'} \tag{3.5}$$

and we can install them into (3.4) to get (lots of steps here)

$$A(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left[\int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \right]$$

$$= \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \int_{-\infty}^{\infty} dt' \left\{ (1/2\pi) \int_{-\infty}^{\infty} d\omega''\, B(\omega'')\, e^{+i\omega''(t-t')} \right\} \left\{ (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, e^{+i\omega' t'} \right\}$$

$$= (1/2\pi)^2 \int_{-\infty}^{\infty} d\omega'' B(\omega'') \int_{-\infty}^{\infty} d\omega' C(\omega') \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \int_{-\infty}^{\infty} dt'\ e^{+i\omega''(t-t')} e^{+i\omega' t'}$$

$$= (1/2\pi)^2 \int_{-\infty}^{\infty} d\omega'' B(\omega'') \int_{-\infty}^{\infty} d\omega' C(\omega') \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\ e^{i(\omega''-\omega)t} e^{i(\omega'-\omega'')t'}$$

$$= (1/2\pi)^2 \int_{-\infty}^{\infty} d\omega'' B(\omega'') \int_{-\infty}^{\infty} d\omega' C(\omega') \left[\int_{-\infty}^{\infty} dt\, e^{i(\omega''-\omega)t} \right] \left[\int_{-\infty}^{\infty} dt'\ e^{i(\omega'-\omega'')t'} \right]$$

$$= (1/2\pi)^2 \int_{-\infty}^{\infty} d\omega'' B(\omega'') \int_{-\infty}^{\infty} d\omega' C(\omega') [2\pi\, \delta(\omega''-\omega)] [2\pi\delta(\omega'-\omega'')]$$

$$= \int_{-\infty}^{\infty} d\omega'' B(\omega'')\, \delta(\omega''-\omega) \left[\int_{-\infty}^{\infty} d\omega' C(\omega')\, \delta(\omega'-\omega'') \right] = \int_{-\infty}^{\infty} d\omega'' B(\omega'')\, \delta(\omega''-\omega) [\, C(\omega'') \,]$$

$$= \int_{-\infty}^{\infty} d\omega'' B(\omega'')C(\omega'')\, \delta(\omega''-\omega) = B(\omega)\, C(\omega)\ .$$

Thus, we have proven that

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Rightarrow \qquad A(\omega) = B(\omega)\, C(\omega)\ .$$

Using the very same method, one can show that ⇐ is also true, and we end up with this very important theorem:

<u>The Convolution Theorem:</u>

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega)\, C(\omega) \qquad (3.6)$$

The significance of this result cannot be overstated. It says that, whereas the relationship between a,b,c might be complicated in the time domain as shown on the left, that complication goes away in the frequency domain on the right, where we have a simple product of functions A = BC. One says that the Fourier Integral Transform "diagonalizes" the convolution integral.

Again using the same method of proof, one can obtain this corresponding theorem:

$$A(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega'\, B(\omega-\omega')C(\omega') \qquad \Leftrightarrow \qquad a(t) = b(t)\, c(t) \qquad (3.7)$$

The extra (1/2π) factor arises because we started with (1.1) and (1.2) which are not symmetric.

Comments

(1) Dimensions. In the convolution equation in (3.6), we shall think of a(t) and c(t) as having the same dimensional units we generically call V, because we are going to think of this equation as being a "filter" where c(t) is the input, a(t) is the output, and b(t) is the "filter kernel". In the examples of this document, we shall take a(t) and c(t) to be dimensionless, but in some application one might add a dimension of "volts" or "amperes" to the functions a(t) and c(t). Looking at (3.6), we find that if a(t) and c(t) are dimensionless or have the same dimensions, then b(t) must have dimensions of inverse time. Looking then at (1.1), we see that A(ω) and C(ω) have dimensions of time, whereas B(ω) is dimensionless. This then is how the dimensions work out in A(ω) = B(ω) C(ω).

(2) Operators. In the language of linear operators, one can regard the functions a and c in (3.6) as vectors in an infinite dimensional vector space of functions, and then the left equation of (3.6) is a "matrix equation" which says $\mathbf{a} = \boldsymbol{\mathcal{b}}\mathbf{c}$ where $\boldsymbol{\mathcal{b}}$ is a linear operator. Specifically, it is an "integral operator". Operator $\boldsymbol{\mathcal{b}}$ acts on vector $\mathbf{c}$ to produce vector $\mathbf{a}$. One could think of (3.6) as a matrix equation $a_t = \Sigma_{t'}\, \mathcal{b}_{tt'} c_{t'}$ where the continuous time variables act as indices. If we similarly write the right side of (3.6) as $A_\omega = \Sigma_{\omega'}\, \mathcal{B}_{\omega\omega'} C_{\omega'}$ then we find that the matrix $\mathcal{B}_{\omega\omega'} = \delta_{\omega,\omega'} B_\omega$ so matrix $\mathcal{B}$ is "diagonal", hence the term "diagonalization". We shall not pursue this language much, but make the reader aware of this interpretation. For more on this subject see Stakgold Chapter 3 on linear integral equations.

(3) Groups. In the more general theory of Fourier Analysis on groups, the "projection" and "expansion" have this form, analogous to (1.1) and (1.2), where σ plays the role of ω and g the role of t,

$$F^\sigma_{kk'} = \int dg\, f(g)\, D^\sigma_{kk'}(g^{-1}) \qquad \text{// projection, transform}$$

$$f(g) = \Sigma_\sigma\, d^\sigma\, \Sigma_{k,k'}\, F^\sigma_{kk'}\, D^\sigma_{k'k}(g) \;=\; \Sigma_\sigma\, d^\sigma\, \mathrm{tr}[F^\sigma D^\sigma(g)] \qquad \text{// expansion, inverse transform}$$

where in the last line tr means trace and F and D are regarded as square matrices. Here g refers to a set of group variables like Euler angles ψ,θ,φ for the rotation group. The functions $D^\sigma_{k'k}(g)$ are the "matrix representations" of the group which have some dimension d^σ. To say that a set of matrices forms a group representation means that, when multiplied, the matrices which represent group elements have the same multiplicative property had by the abstract group elements themselves,

$$\Sigma_{k''}\, D^\sigma(g_1)_{kk''}\, D^\sigma(g_2)_{k''k'} = D^\sigma(g_3)_{kk'} \qquad \text{or} \qquad D^\sigma(g_1)\, D^\sigma(g_2) = D^\sigma(g_3) \qquad \text{"group property"}$$

where $g_1 g_2 = g_3$. Quantity dg is the "invariant measure" on the group which is dψd(cosθ)dφ for the rotation group. In our simple Fourier Transform case, we have dg = dt, the group is the group of translations along the time axis, and the matrix representations have dimension $d^\sigma = 1$ and are thus 1x1 matrices, namely, $e^{-i\omega t}$. The group property shown above is just $e^{-\omega_1 t}\, e^{-\omega_2 t} = e^{-(\omega_1+\omega_2)t}$.

In the general case, the convolution equation and its diagonalization are given by this generalized convolution theorem,

$$a(g) = \int dg_1\, b(g_1^{-1}g)c(g_1) \qquad \Leftrightarrow \qquad A^\sigma{}_{kk'} = \Sigma_{k''}\, B^\sigma{}_{kk''}\, C^\sigma{}_{k''k'}$$

where the dg_1 integral is over the entire parameter space of the group. The derivation of this theorem makes use of the group property shown above and the fact that $dg_3 = d(g_1g_2) = dg_1$ when the integration is over the full group space, just as in the simple case $dt_3 = d(t_1+t_2) = dt_1$. This is why dg is referred to as the "invariant" measure.

In the case of one-dimensional representations, the convolution theorem says $A^\sigma = B^\sigma C^\sigma$ which is our $A(\omega) = B(\omega)C(\omega)$ with $\sigma = \omega$. Despite the sum on k", the equation on the right is said to be "diagonalized" because it is true separately for each value of the label σ. If one writes $B^\sigma{}_{kk''} = \delta_{\sigma,\sigma''} B_{\sigma k,\sigma''k''}$, then the matrix $B_{\sigma k,\sigma''k''}$ is diagonal in the sense that it is mostly zero but has square matrices of size d^σ x d^σ on its diagonal ("block diagonal form"). For more on the subject of Fourier analysis on groups, see Hermann.

4. Applications of the Convolution Theorem

This section is included because books often do not make the connection between the convolution theorem, Green's Functions, and the real world of everyday electronics. Often too this discussion is presented in the language of Laplace Transforms, so here we work in terms of the above Fourier Transform. We shall state the general case, then do specific examples.

(a) General case

The real world seems to be described by linear differential equations. Here is a general form:

$$L_t\, u(t) = f(t) \qquad (4.1)$$

where L_t contains perhaps first and second order differential operators d/dt and d^2/dt^2. One would like to solve this equation for u, given some driving function f. It would be nice if one could find some operator that is the inverse of L_t and apply it to both sides of (4.1); the problem would then be solved. This is exactly what we are going to do. We first define a related equation as follows,

$$L_t\, g(t-t') = \delta(t-t') \,. \qquad (4.2)$$

Here g(t-t') is the "impulse response" of the differential equation to the driving impulse term δ(t-t'). If we can solve (4.2) for g, then we know a solution to (4.1) for u(t) in terms of f and g, namely,

$$u(t) = \int_{-\infty}^{\infty} dt'\, g(t-t')\, f(t') \,. \qquad (4.3)$$

In general one can add to this "particular" solution any solution of (4.1) with f(t) = 0. These extra homogeneous solutions can be tailored to meet required boundary conditions.

<u>Proof:</u>

$$L_t\, u(t) = L_t \left\{ \int_{-\infty}^{\infty} dt'\, g(t-t')\, f(t') \right\} = \int_{-\infty}^{\infty} dt'\, [L_t\, g(t-t')]\, f(t') = \int_{-\infty}^{\infty} dt'\, [\delta(t-t')\,]\, f(t') = f(t) \,.$$

The function g is called the "Green's Function", "propagator", or "kernel" of L_t. In (4.3) one is applying an integral operator $G = \int g$ to function f to get function u, so u = Gf. Looking at (4.1), this integral operator G must in some sense be the inverse of the differential operator L_t.

Now we come to the main point: equation (4.3) is a convolution equation of the form (3.6)! Therefore, we can write (4.3) in the ω-domain as follows:

$$U(\omega) = G(\omega)\, F(\omega) \,. \qquad (4.4)$$

(b) A specific example: the RC filter section

Consider a simple unloaded RC filter section with input voltage $v_i(t)$ and output voltage $v_o(t)$,

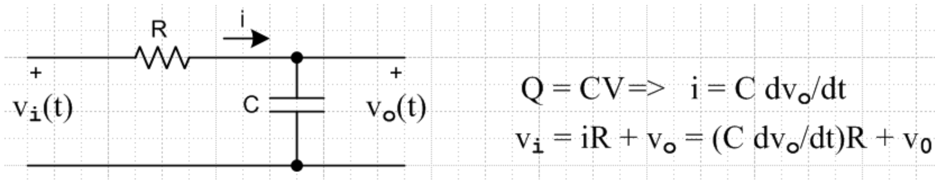


Fig 4.1

Here is the differential equation, derived on the right above,

$$[\,RC\, d/dt + 1]\, v_o(t) = v_i(t) \quad . \tag{4.5}$$

Define the Green's Function g(t) by

$$[\,RC\, d/dt + 1]\, g(t) = \delta(t) \;. \tag{4.6}$$

Then the solution to (4.5) is this:

$$v_o(t) = \int_{-\infty}^{\infty} dt'\, g(t-t')\, v_i(t') \quad . \tag{4.7}$$

This has the convolution form, so in the frequency domain we get

$$V_o(\omega) = G(\omega)\, V_i(\omega). \tag{4.8}$$

Sometimes this is called "filter theory", where G(ω) is the "transfer function" of the filter -- in our case a simple RC filter. If we expand g(t) as in (1.2) and δ(t) as in (2.1) then (4.6) says

$$[\,RC\, d/dt + 1]\, (1/2\pi) \int_{-\infty}^{\infty} d\omega\, G(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}$$

or

$$\int_{-\infty}^{\infty} d\omega\, G(\omega)\, [\,RC\, d/dt + 1]\, e^{+i\omega t} = \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t}$$

or

$$\int_{-\infty}^{\infty} d\omega\, G(\omega)\, [\,RC\, (i\omega) + 1]\, e^{+i\omega t} = \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} \;.$$

Since the basis functions $e^{i\omega t}$ form a complete set on the interval (-∞,∞), we conclude that

$$G(\omega)\, [\,RC\, (i\omega) + 1] = 1$$

or

$$G(\omega) = 1/\, [\,1 + i\omega RC]$$

or

$$G(\omega) = (1/i\omega C) \,/\, [(1/i\omega C) + R] = (-iX_c)/\, [\,R + (-\,iX_c)] \qquad // \; X_C = \text{capacitive reactance} = (\omega C)^{-1}$$

or

$G(\omega) = Z_C/(R+Z_C)$. // $Z_C = -i X_C$

In the frequency domain, we see $G(\omega)$ as the output of a simple voltage divider where one element has real impedance R and the other imaginary impedance Z_C.

The above series of steps shows that in the frequency domain, one can replace d/dt by $i\omega$.

Let $\tau \equiv RC$ and compute g(t) using (1.2),

$$g(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, G(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} / [\, 1 + i\omega\tau]$$

$$= (1/2\pi i\tau) \int_{-\infty}^{\infty} d\omega\, e^{+i\omega t} / [\, \omega - i/\tau] \ .$$

Thinking of this as a contour integral,

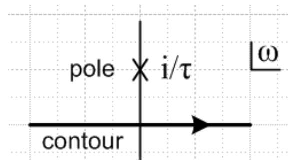


Fig 4.2

For t > 0 we can close in the upper half plane and pick up the residue of the pole sitting at $\omega = i/\tau$ to get

$$g(t) = (1/2\pi i\tau)\, 2\pi i\, e^{i(i/\tau)t} = (1/\tau)\, e^{-t/\tau} = (1/RC)\, e^{-t/RC} \ .$$

For t < 0 we close instead in the lower half plane and pick up nothing, so the result is 0. Thus

$$g(t) = (1/RC)\, e^{-t/RC}\, \theta(t)$$

where $\theta(t)$ is the Heaviside step function. To summarize, the transfer function $G(\omega)$ and its time-domain Green's Function g(t) are:

$$G(\omega) = 1/(\, 1 + i\omega RC) = Z_C/(\, R + Z_C) \qquad (4.9)$$

$$g(t) = (1/RC)\, e^{-(t/RC)}\, \theta(t) \ . \qquad (4.10)$$

If $v_i(t) = \delta(t)$, then from (1.1) we have $V_i(\omega) = 1$. In this case, $V_o(\omega) = G(\omega)\, 1$ and it must be that $v_0(t) = g(t)$. Thus, one always interprets g(t) as the impulse response of the filter. In this case, it is of course a simple decaying exponential. One then interprets $\theta(t)$ as saying that the impulse response only propagates forward in time, never backward ("causality").

We conclude this section by writing out (4.7), which shows the time domain solution of our simple RC filter:

$$v_o(t) = \int_{-\infty}^{\infty} dt'\, g(t-t')\, v_i(t') \quad = (1/RC) \int_{-\infty}^{\infty} dt'\, e^{-(t-t')/RC}\, \theta(t-t')\, v_i(t')$$

$$= (1/RC) \int_{-\infty}^{t} dt'\, e^{-(t-t')/RC}\, v_i(t') \quad . \tag{4.11}$$

This says that the present response of the system at time t is the cumulative result of the impulse responses at all past times, weighted by the value of the input function $v_i(t')$. In other disciplines, the expression $(1/RC)e^{-(t-t')/RC} = g(t-t')$ is called a propagator, since it describes exactly how the voltage amplitude $v_i(t')$ at some past time propagates into $v_o(t)$ at a some future time. This terminology is more useful when the integral operator has more than one variable. If dt' were replaced by dt' $d^3\mathbf{x}'$, then an equation like (4.11) would perhaps describe how a wave propagates through 3D space. The result is then the sum of "scattering" at all t' in the past, and all positions **x'** in space. In our example, there is no spatial aspect, and the output voltage is just the input voltage scattered off the RC filter at all times in the past.

(c) An even simpler example: $L_t = (d/dt)$

In this section, we use the same equation numbers as in section b above, but add label c, as in $(4.9)_c$.

Let's start by considering $L'_t = RC\,(d/dt)$. If RC is regarded as very large, RC >> 1, then we may take over the results (4.9) and (4.10) of the previous section as follows:

$$G'(\omega) = 1/(i\omega RC) \qquad \text{for} \qquad L'_t = RC\,(d/dt)$$

$$g'(t) = (1/RC)\, \theta(t) \quad .$$

Then if we rescale so that $L_t = (1/RC)\, L'_t = (d/dt)$, we just multiply the above results by RC to get

$$G(\omega) = 1/(i\omega) \qquad \text{for} \qquad L_t = (d/dt). \tag*{$(4.9)_c$}$$

$$g(t) = \theta(t) \quad . \tag*{$(4.10)_c$}$$

Our starting differential equation is

$$[d/dt]\, v_o(t) = v_i(t) \quad . \tag*{$(4.5)_c$}$$

Define the Green's Function g(t) by

$$[d/dt]\, g(t) = \delta(t) \quad . \tag*{$(4.6)_c$}$$

Then the solution to $(4.5)_c$ is this:

$$v_o(t) = \int_{-\infty}^{\infty} dt'\, g(t-t')\, v_i(t') \quad . \tag*{$(4.7)_c$}$$

This has the convolution form, so in the frequency domain we get

$$V_o(\omega) = G(\omega)\, V_i(\omega) = [\, 1/(i\omega)]\ V_i(\omega) \ . \qquad (4.8)_c$$

Inserting g(t-t') = θ(t-t') from $(4.10)_c$ we get

$$v_o(t) = \int_{-\infty}^{\infty} dt'\, g(t-t')\, v_i(t') \ = \int_{-\infty}^{t} dt'\, v_i(t') \ . \qquad (4.11)_c$$

We can differentiate this result to obtain the starting equation $(4.5)_c$.

In this case, the time propagator is simply g(t-t') = θ(t - t'). The forward propagator amplitude is just 1 regardless of how far t and t' are separated, and the propagator is 0 if it tries to send something backwards in time. In other words, causality is built into this propagator, and this was also the case for the RC filter (4.10). Physically, this example is an RC filter with a *very* long time constant, so basically all effects from the recent past propagate to the present with no attenuation. The capacitor is an integrator, just as one uses in an operational-amplifier-based analog computer design.

There are some subtleties involving the transform pair G(ω) = 1/(iω) and g(t) = θ(t) which have been swept under the rug in the last few paragraphs, but which are laid bare in Appendix C.

5. Fourier Integral Transform Conventions

(a) Sign of Phase. The Fourier Transform (1.1) and (1.2) is also true if one replaces i with -i in both equations. This follows trivially from (2.1). EE people usually think of the fundamental "spectral component" time dependence as $e^{+i\omega t}$ and cos(ωt), so they want to see $e^{+i\omega t}$ in the expansion (1.2). Physics people who are often pondering plane waves described by exp[+i(**k•r** - ωt)] or cos(**k•r** - ωt) want to see $e^{-i\omega t}$ in the expansion (1.2). We have chosen to use the EE convention. If you want to use the physics convention, you must replace all our i by -i, and also Im[ ] by -Im[ ]. The physics convention is used, for example, by Stakgold Vol. II page 23 equation (5.32), a source we sometimes quote below.

(b) j Versus i. EE texts favor j, physics texts always use i, which is of course the true historical symbol for $\sqrt{-1}$. The reason is that EE people deal with lumped circuits containing currents labeled "i", whereas physicists deal with Maxwell's equations which contain current density "j". Each discipline chooses its symbol for $\sqrt{-1}$ to minimize confusion with these other symbols. We shall use i.

(c) Allocation of 2π. Our convention has been to put the factor of (1/2π) into the inversion formula (1.2), and to have no factor at all in the transform formula (1.1). We shall describe our motivations for doing this below.

Sometimes books put a $1/\sqrt{2\pi}$ factor in the transform (1.1), which causes the appearance of an identical $1/\sqrt{2\pi}$ factor in equation (1.2). This has the advantage of making the two equations completely symmetrical, and reminds us that there is complete symmetry between the conjugate variables t and ω. We have chosen not to do this in our presentation.

And of course the world would not be complete if some people did not prefer to put a 1/2π into the expansion equation (1.1), and have none of it in (1.2).

In general, the product of the two factors must be 1/2π. This is simply due to the 2π factor sitting on the right of (2.1). The main reason we choose to put the factor entirely in (1.2) is the following. Suppose we have a constant k≠1 on the right side of (1.1). Then the transform X(ω) so defined is scaled differently than our X(ω). If we rescale all terms in the ω-plane part of the convolution theorem (3.6), we must end up with an extra factor of k hanging around in the new version of the right side of (3.6). The other alternative is to add a k factor into the definition of the convolution integral (3.1). Neither is very nice, and there is a lot of history behind (3.6) as written. This is why we have done our 2π factors as shown above.

(d) Comments. The conventions discussed above have no real physical significance, they just lead to different definitions of X(ω), so there are slight variations in (1.1) and (1.2). It is important to at least adopt *some* convention so one knows what one is talking about. A potential problem comes when one tries to look up something in a table or handbook; one may be off by a factor of 2π or $\sqrt{2\pi}$ if one is not clear on the conventions (attention people sending spacecraft to planets). The conventions we have adopted are consistent with 33.7,8 of the 1968 Schaum's Mathematical Handbook (now 4th Ed. 2012), and also with a 1967 printing of the Fourth Edition ITT Reference Data for Radio Engineers (now 9th Ed. 2001). There must be something good about these two publications since they are both alive and well after half a century.

One other small convention detail is that, with our adopted phase convention, spectra X(ω) are normally analytic in the lower half ω plane and have poles in the upper half plane. Use of the other phase sign results in X(ω) which are analytic in the upper half ω plane and have poles in the lower half plane, since in effect the entire ω plane is reflected in the real ω axis by a change of sign phase. This affects the form of dispersion relations, as we shall see in Chapter 5.

6. The Generalized Fourier Integral Transform and the Laplace Transform $\mathcal{X}$(s)

In the discussion above, the Fourier Integral Transform spectrum X(ω) is defined for ω real and for x(t) being L_1 integrable. One can show that the idea of the Fourier Transform can be extended to allow for x(t) which are not L_1 integrable, provided one thinks of ω as a complex variable, and one thinks of the inversion integral contour of (1.2) as being a horizontal line in the complex ω plane which runs below any possible singularities of X(ω). In this extension of the Fourier Integral Transform, one must use single-sided functions, and one usually deals with right-sided (causal) functions which vanish for t < 0. Such a function has the general form x(t) = θ(t)f(t). As an example, suppose

$$x(t) = \theta(t)e^{\alpha t} . \qquad (6.1)$$

In this case, (1.1) says

$$X(\omega) = \int_0^\infty dt\, e^{\alpha t}\, e^{-i\omega t} = \int_0^\infty dt\, e^{(\alpha-i\omega)t} = \frac{-1}{\alpha-i\omega} = \frac{-i}{\omega-(-i\alpha)} \qquad (6.2)$$

which has a pole at ω = -iα. The integral converges because we assume that ω has a sufficiently large negative imaginary part (perhaps -ic) to make it converge. The inversion formula is then

$$x(t) = (1/2\pi) \int_{-ci-\infty}^{-ci+\infty} d\omega\, X(\omega)\, e^{+i\omega t} = (-i/2\pi) \int_{-ci-\infty}^{-ci+\infty} d\omega \frac{e^{+i\omega t}}{\omega-(-i\alpha)} \qquad (6.3)$$

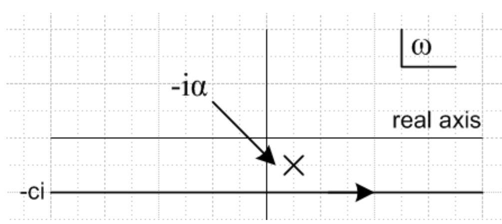


Fig 6.1

where we position the contour at -ci which we assume lies below the pole. For t < 0, we close the contour downward, $e^{+i\omega t}$ decays, the great circle makes no contribution, and we recover that fact that x(t) = 0 for t < 0. For t > 0 we close upward and wrap the pole to get

$$x(t) = (-i/2\pi)\ 2\pi i\, e^{i(-i\alpha)t} = e^{\alpha t}$$

which of course is the desired result.

So our **generalized Fourier Integral Transform** may be stated as :

$$X(\omega) = \int_0^\infty dt\ x(t)\ e^{-i\omega t} \qquad \text{projection = transform} \qquad (6.4)$$

$$x(t) = (1/2\pi) \int_{-ci-\infty}^{-ci+\infty} d\omega\ X(\omega)\ e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (6.5)$$

where -ci lies below all singularities of $X(\omega)$.

For a detailed discussion of this subject, see Stakgold Vol 2 pp 23-28. Since Stakgold uses the opposite phasor sign in his definition of the Fourier Transform, the ω plane contour for him is raised up so it runs above all poles of $X(\omega)$, which he calls $x\hat{}(\omega)$. Stakgold is interested in the Fourier Transform of a distribution, but in these pages he talks only about functions.

If we now change variables from ω to $s = i\omega$, the above generalized Fourier transform becomes

$$X(s/i) = \int_0^\infty dt\ x(t)\ e^{-st} \qquad (6.6)$$

$$x(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} ds\ X(s/i)\ e^{+is} \qquad (6.7)$$

where the contour in the s-plane is as shown here, lying to the right of all singularities of X(s/i),

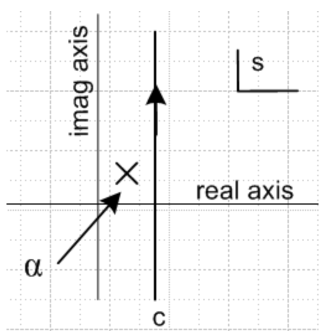


Fig 6.2

We rotated the previous picture 90° counterclockwise to get the s-plane picture. If we now define

$$\mathcal{X}(s) \equiv \mathcal{L}[x(t), s] \equiv X(s/i) \qquad (6.8)$$

the transform becomes

$$\mathcal{X}(s) = \int_0^\infty dt\ x(t)\ e^{-st} \qquad (6.9)$$

$$x(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} ds\ \mathcal{X}(s)\ e^{+is} \qquad (6.10)$$

which is the **Laplace Transform** and its inverse. The inversion contour runs to the right of all singularities in $\mathcal{X}(s)$.

Here then is our conclusion: for the set of functions x(t) which are "causal", like our Green's Function propagator g(t) discussed above, and which therefore vanish at negative time, we can make an exact identification between the Laplace Transform $\mathcal{X}(s)$ and the generalized Fourier Transform X(ω) evaluated at ω = s/i. If we think of s = real, then we are "analytically continuing" the function X(ω) off its real ω axis. If we think of ω as real, then we are analytically continuing the Laplace Transform to imaginary s.

For non-causal functions x(t), the Fourier and Laplace Transforms do not have this simple relationship. Of course, when considering some general function x(t), we can easily make it causal "by fiat" by simply multiplying it by θ(t). In this case, our association holds all the time,

$$\mathcal{L}[\theta(t)x(t), s] = X(s/i) \qquad\qquad X(\omega) = \mathcal{L}[\theta(t)x(t), i\omega] \ . \tag{6.11}$$

This lets us make use of extensive tables of Laplace Transforms to look up X(ω) for given x(t), and lets us also understand that the "general properties" of Laplace Transforms also apply to the Fourier Transform, with the appropriate replacement s = iω. A very large table (~ 100 pages) of Laplace Transforms appears in the Bateman Manuscript Project Vol. 4 (see Erdelyi. *et. al.*).

Example: The Laplace Transform of x(t) = e^{at} is 1/(s-a), and a trivial "property" of the Laplace Transform is that kx(t) maps into k $\mathcal{L}$ [x(t),s] (the transform is linear). Thus, for our Green's Function of (4.10), using a = -1/RC,

$$\mathcal{L}[\,g(t), s] = \mathcal{L}[\,(1/RC)\, e^{-t/(RC)}\, \theta(t)\, , s] = (1/RC)\,[1\,/\,(s + (1/RC))] = 1/(sRC + 1)\ . \tag{6.12}$$

Thus we would conclude from (6.12) that

$$G(\omega) = 1/(i\omega RC + 1) \tag{6.13}$$

which agrees with (4.9) above.

7. Reflection Rules

For general complex x(t), the definition (1.1) shows that

$$x(t) \leftrightarrow X(\omega) \quad \Leftrightarrow \quad x(-t) \leftrightarrow X(-\omega) \qquad \text{// complex } x(t) \qquad (7.1)$$

$$x(t) \leftrightarrow X(\omega) \quad \Leftrightarrow \quad x(-t)^* \leftrightarrow X(\omega)^* \qquad \text{// complex } x(t) \qquad (7.2)$$

This notation, used later, means that if x(t) has spectrum X(ω), then x(-t) has spectrum X(-ω) and similarly for the second line.

<u>Proofs:</u>

$$Y(\omega) = \int_{-\infty}^{\infty} dt\, y(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, x(-t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, x(t)\, e^{+i\omega t} = X(-\omega)$$

$$Y(\omega) = \int_{-\infty}^{\infty} dt\, y(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, x(-t)^*\, e^{-i\omega t} = [\int_{-\infty}^{\infty} dt\, x(-t)\, e^{+i\omega t}]^*$$
$$= [\int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t}]^* = X(\omega)^*$$

If we do assume that x(t) is real (for example, a voltage or current in a real circuit), then from (1.1) the following fact follows at once (* means complex conjugation),

$$X(-\omega) = [X(\omega)]^* \quad . \qquad \text{// if } x(t) \text{ is real} \qquad (7.3)$$

Thus, one can think of the mysterious negative frequency spectral components of a real function x(t) as simply being defined in this manner in terms of the positive spectral components. Note that X(ω) is in general complex, even if x(t) is real, because exp(-iωt) is complex in (1.1).

Similarly, (1.2) says that

$$x(-t) = [x(t)]^* \qquad \text{// if } X(\omega) \text{ is real} \qquad (7.4)$$

A function having the property f(-x) = f*(x) is called a **Hermitian function**. So we have shown that if x(t) is real, then X(ω) is Hermitian, and if X(ω) is real, then x(t) is Hermitian.

If x(t) is real, then (7.3) implies

$$| X(-\omega)|^2 = | X(\omega)|^2 \qquad (7.5)$$

and this is why, when dealing with spectral densities (as we shall below), many authors simply reflect the left half of the spectrum to the right side which doubles the right side. This must be done carefully if the spectrum includes a δ(ω) term: the folded spectrum for such a term gets a factor of 1/2. In general we shall not use such folded spectra.

Two final rules are these:

$$x(t) = \text{even in } t \qquad \Leftrightarrow \qquad X(\omega) = \text{even in } \omega \tag{7.6}$$

$$x(t) = \text{real and even in } t \qquad \Leftrightarrow \qquad X(\omega) = \text{real and even in } \omega \tag{7.7}$$

Proof of both together:

$\Rightarrow$ $X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\, x(t)\ [\cos(\omega t) - i\sin(\omega t)] = \int_{-\infty}^{\infty} dt\, x(t) \cos(\omega t)$ = even in ω

and if in addition x(t) is real, then $\int_{-\infty}^{\infty} dt\, x(t) \cos(\omega t)$ is even in ω *and* real

$\Leftarrow$ $x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega) \cos(\omega t)$ = even in t

and if in addition X(ω) is real, then $(1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega) \cos(\omega t)$ is even in t *and* real

8. Three simple examples of spectra

(a) The spectrum of x(t) = 1 :

In this example, x(t) is a constant over all time. Using (1.1) and (2.1), we find:

$$x(t) = 1 \qquad\qquad X(\omega) = 2\pi\, \delta(\omega)\ . \tag{8.1}$$

This comes as no surprise. For a DC signal, all the energy is concentrated at zero frequency. Of course x(t) = 1 does not respect the requirement $\int_{-\infty}^{\infty} dt\, |x(t)| < 0$, which is why the spectrum is a distribution.

(b) The spectrum of $x(t) = \delta(t - t_1)$:

Here x(t) is an infinitely narrow pulse of area 1, positioned at $t = t_1$. Using (1.1), we get the following Fourier spectrum:

$$x(t) = \delta(t - t_1) \qquad X(\omega) = e^{-i\omega t_1} \quad . \tag{8.2}$$

The spectrum X(ω) has a constant magnitude 1 for all ω, out to infinite frequency. For such a pulse at t=0,

$$x(t) = \delta(t) \qquad X(\omega) = 1 \tag{8.3}$$

and here the phase is constant. This result is (8.1) with ω ↔ t and the constant adjusted due to the asymmetry of the transform in our adopted convention. From Appendix C we quote this general rule

$$\text{FT of } x(t) = X(\omega) \quad \Leftrightarrow \quad \text{FT of } X(t) = 2\pi x(-\omega) \tag{C.5}$$

which, when applied to (8.3), gives (8.1) since $\delta(-\omega) = \delta(\omega)$.

(c) The spectrum of $x(t) = \theta(t)$:

The regular Fourier Integral Transform spectrum of the Heaviside step function θ(t) is the somewhat peculiar first line following, whereas the generalized Fourier Integral Transform gives the second line

$$x(t) = \theta(t) \qquad X(\omega) = \text{"}\frac{1}{i\omega}\text{"} = \frac{1}{i\omega+\varepsilon} = \frac{1}{i}\frac{1}{\omega - i\varepsilon}$$

$$X(\omega) = \frac{1}{i\omega} \qquad \text{// generalized Fourier Integral Transform of (6.4)} \tag{8.4}$$

which we now explain. Like x(t) = 1, the Heaviside θ(t) is also not in the class of functions for which the Fourier Transform is defined ($\int_{-\infty}^{\infty} dt\, |x(t)| < 0$). We bring θ(t) into the acceptable class by replacing θ by θ_ε where,

$$\theta_\varepsilon(t) \equiv \begin{cases} e^{-\varepsilon t} & t > 0 \\ 0 & t < 0 \end{cases} \qquad \text{for some very small } \varepsilon > 0 \quad . \tag{8.5}$$

Then

$$X_\varepsilon(\omega) = \int_{-\infty}^{\infty} dt\, \theta_\varepsilon(t)\, e^{-i\omega t} = \int_0^{\infty} dt\, e^{-\varepsilon t}\, e^{-i\omega t} = \frac{1}{i\omega+\varepsilon}$$

and then $X(\omega) = \lim_{\varepsilon\to 0} X_\varepsilon(\omega) = (1/i\omega)$. *But* we really have to think of (1/iω) as meaning the limit of $\frac{1}{i\omega+\varepsilon}$. To see why, we now compute x(t) from the inversion formula,

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega \frac{1}{i\omega+\varepsilon} e^{+i\omega t} = (1/2\pi i) \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon} e^{+i\omega t} . \qquad \text{pole at } \omega = +i\varepsilon$$

For $t < 0$, close the ω contour down and get 0 since the great circle vanishes. For $t > 0$ close up and pick up the pole reside to get $x(t) = e^{-\varepsilon t}$. Thus we have recovered (8.5). For $t = 0$, we let the pole move to the real axis from above and deflect the contour down

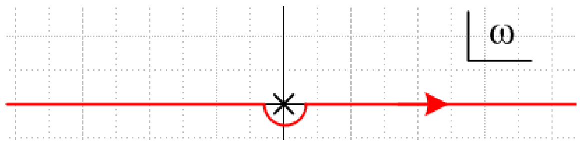


Fig 8.1

In the limit the contour is shrunk around the pole, the contributions from $(-\infty,0)$ and $(0,\infty)$ cancel. These two terms are known as a principle value integral and we have

$$\text{PV} \int_{-\infty}^{\infty} d\omega\, (1/\omega) \equiv ⨍_{-\infty}^{\infty} d\omega\, (1/\omega) = 0$$

since the left and right sides cancel (even range integral of an odd function). All that is left is the half turn around the pole which picks up half the residue at the pole (see Appendix C) so the result is then

$$x(0) = (1/2\pi i) \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - i\varepsilon} = (1/2\pi i)\ (1/2)\ (2\pi i * 1) = 1/2$$

and we obtain the fact that $\theta(0) = 1/2$ as was shown in Fig 1.1.

If we apply our upcoming differentiation rule (11.1) [which is to multiply by $i\omega$] we find that

$$\theta(t) \leftrightarrow \frac{1}{i\omega+\varepsilon} \qquad => \qquad \delta(t) = d\theta(t)/dt \leftrightarrow i\omega \frac{1}{i\omega+\varepsilon} = 1$$

which agrees with (8.3) above.

Using the *generalized* Fourier Integral Transform stated in (6.4) and (6.5), we can regard $X(\omega) = 1/(i\omega)$ without all the ε business since the ω recovery contour in (6.5) runs below all singularities in the ω plane, which contour, when deformed up, gives Fig 8.1 and all the results quoted above. Applying our Laplace equivalence notion (6.8), we would predict from $X(\omega) = 1/(i\omega)$ that

$$\mathcal{L}[\theta(t), s] = \mathcal{X}(s) = X(s/i) = \frac{1}{i(s/i)} = \frac{1}{s}$$

which is in agreement with any Laplace table.

The above examples are treated in more detail in Appendix C where the pf pseudofunction is introduced and the Pole Avoidance Rule is derived and then used.

9. Spectrum of an isolated square pulse

Consider a positive square pulse of amplitude A and width τ which is centered at t=0. We can represent this using the Heaviside step function,

$$x(t) = A\,[\,\theta(t+\tau/2) - \theta(t-\tau/2)\,] \qquad // = A\,\mathrm{rect}(t/\tau)$$

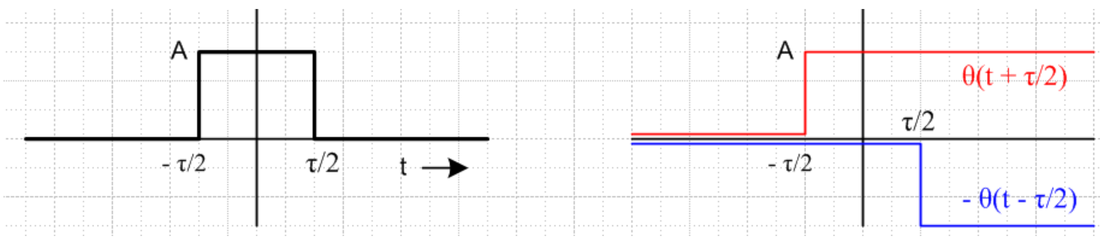


Fig 9.1

This "square" pulse is in general rectangular and we often refer to it as a box-shaped pulse.

Apply (1.1) to get the spectrum of this pulse,

$$X(\omega) = \int_{-\infty}^{\infty} dt\; x(t)\, e^{-i\omega t} = A \int_{-\infty}^{\infty} dt\, \{\,[\,\theta(t+\tau/2) - \theta(t-\tau/2)\,]\}\, e^{-i\omega t}$$

$$= A\,[\int_{-\tau/2}^{\infty} - \int_{\tau/2}^{\infty}]\, dt\, e^{-i\omega t} = A \int_{-\tau/2}^{\tau/2} dt\, e^{-i\omega t} = A \int_{-\tau/2}^{\tau/2} dt\, \cos(\omega t) \qquad // \sin = \text{odd}$$

$$= 2A \int_{0}^{\tau/2} dt\, \cos(\omega t) = 2A\, \sin(\omega\tau/2)/\omega = (A\tau)\,[\sin(\omega\tau/2)]\,/\,(\omega\tau/2) = (A\tau)\,\mathrm{sinc}(\omega\tau/2)$$

where we use the definition $\mathrm{sinc}(x) \equiv \sin(x)/x$ (there are other definitions). To summarize:

$$x(t) = A\,[\,\theta(t+\tau/2) - \theta(t-\tau/2)\,] \tag{9.1}$$

$$X(\omega) = (A\tau)\,\mathrm{sinc}(\omega\tau/2)\;. \tag{9.2}$$

<u>Observations:</u>

(a) The spectrum is real and even in ω (see (7.7) for why), and it is a continuous function of ω.

(b) Because sinc(-x) = sinc(x), $X(\omega)$ is an even function of ω.

(c) $X(\omega)$ has the shape we are all familiar with.. The positive zeros are at $x = (\omega\tau/2) = n\pi$ for n=1,2,3... The first zero is at $\omega = 2\pi/\tau$ ($f = 1/\tau$). The central peak has height $(A\tau)$. Here is a plot of y = sinc(x),

```
plot(sin(x)/x, x=-20..20,thickness = 2, xtickmarks=10);
```

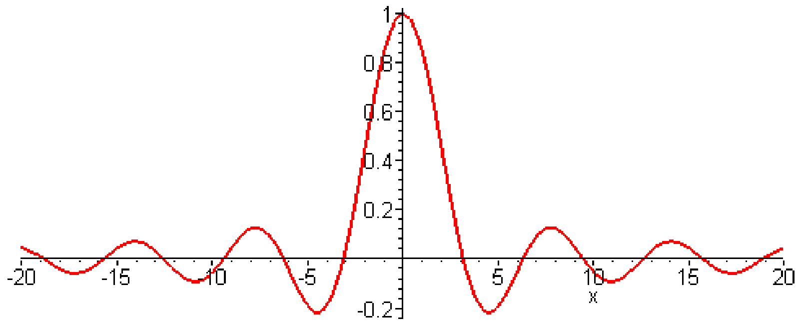


Fig 9.2

(d) Most of the spectral energy is in the central hump, and this represents positive frequency in the range $f=0$ to $f=1/\tau$.

(e) Quantity $(A\tau)$ is the Area under the time-domain pulse. If the pulse is made twice as narrow ($\tau \rightarrow \tau/2$) and twice as high ($A\rightarrow 2A$), this area stays constant, but the first zero of $X(\omega)$ moves out twice as far (as do all zeros), so the spectral width doubles.

(f) In the limit $\tau \rightarrow 0$ with $(A\tau)$ = (Area) = fixed, the square pulse $x(t)$ approaches (Area) $\delta(t)$. Since $\text{sinc}(x) \rightarrow 1$ as $x\rightarrow 0$, we find from (9.2) that $X(\omega)$ = (Area). This is in agreement with (8.3) above. In this limit, the height of the central ω hump stays fixed, and the zeros move out to infinity, as if a small flat portion of the central hump has expanded to fill all ω. A delta function has a "white" spectrum since $X(\omega)$ is a constant for all frequencies.

(g) What about the limit $\tau \rightarrow \infty$? If we take this limit with A = fixed, we are converting our pulse to a constant DC signal $x(t) = A$. In this limit, as long as $\omega \neq 0$, the argument of the sinc function oscillates infinitely fast, giving a function that is zero when averaged over any finite interval. At $\omega = 0$, something singular happens. The result comes out $X(\omega) = 2\pi A\ \delta(\omega)$, in accordance with (8.1) above. In terms of (9.1), in this limit the central hump gets higher and higher, and the zeros all move in toward $\omega = 0$. As these zeros get closer together, the oscillation frequency of the tail of sinc(x) becomes infinite and washes out. To derive $X(\omega) = 2\pi A\ \delta(\omega)$ from (9.2), one can use (A.12)

$$\lim_{B\rightarrow\infty} \delta_4(k,B) \ = \ \lim_{B\rightarrow\infty} \frac{\sin(Bk)}{\pi k} \ = \lim_{B\rightarrow\infty} \frac{B}{\pi}\ \text{sinc}(Bk) \ = \ \delta(k)\ , \qquad \textit{(A.12)}$$

which says, with $k = \omega$ and $B = \tau/2$,

$$\lim_{\tau\rightarrow\infty} \frac{\tau}{2\pi}\ \text{sinc}(\frac{\tau}{2}\,\omega) \ = \ \delta(\omega)$$

so

$$\lim_{\tau\rightarrow\infty} [(A\tau)\ \text{sinc}(\omega\tau/2)] = A\ 2\pi\ \delta(\omega)\ . \qquad (9.3)$$

(h) We can recover the box function from its spectral components using (1.2),

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, (A\tau)\, \mathrm{sinc}(\omega\tau/2)\, e^{+i\omega t}$$

$$= (1/2\pi)\,(A\tau) \int_{-\infty}^{\infty} d\omega(\omega\tau/2)^{-1} \sin(\omega\tau/2)\, e^{+i\omega t}$$

$$= (A\tau\,/2\pi) \int_{-\infty}^{\infty} d\omega(\omega\tau/2)^{-1}\, (1/2i)\, [\, e^{i\omega\tau/2} - e^{-i\omega\tau/2}]\, e^{+i\omega t}$$

$$= (A\tau\,/2\pi)(2/\tau)\,(1/2i) \int_{-\infty}^{\infty} d\omega\, (1/\omega)\, [\, e^{i\omega(t+\tau/2)} - e^{i\omega(t-\tau/2)}]$$

$$= (A\,/2\pi i)\ [\ \int_{-\infty}^{\infty} d\omega\, (1/\omega)\, e^{i\omega(t+\tau/2)} - \int_{-\infty}^{\infty} d\omega\, (1/\omega)\, e^{i\omega(t-\tau/2)}]\ ]\ .$$

Recall from the generalized Fourier Transform discussion of Section 6 that the ω contours run below the pole at $\omega = 0$, so the pole is effectively located at $\omega = +i\varepsilon$. In either integral, if the exponent is positive, the exponential decays on the upper half great circle, so we close the contour up and pick up the pole residue. On the other hand, if the exponent is negative, we close down and pick up nothing. Thus

$$= (A\,/2\pi i)\ \{\ \theta(t+\tau/2)\, 2\pi i\ - \theta(t- \tau/2) 2\pi i\ \}$$

$$= A\,[\ \theta(t+\tau/2) - \theta(t- \tau/2)\ ]$$

which replicates (9.1). A bit more directly, we can compute x(t) on the sides of the box :

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, (A\tau)\, \mathrm{sinc}(\omega\tau/2)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, (A\tau)\, \mathrm{sinc}(\omega\tau/2) \cos(\omega t)$$

so that, using $x = \omega\tau/2$ so $dx = (\tau/2)d\omega$,

$$x(\pm\tau/2) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, (A\tau)\, \mathrm{sinc}(\omega\tau/2) \cos(\pm\omega\tau/2)$$

$$= (A\tau/2\pi)(\tau/2) \int_{-\infty}^{\infty} \mathrm{sinc}(x) \cos(x) = (A\tau/2\pi)(\tau/2)\, \pi/2 = (A/2)\ ,$$

supporting the notion of Section 1 that $x(t) = \lim_{\varepsilon\to 0}\ [\ x(t+\varepsilon) + x(t-\varepsilon)\ ]/2$ at a point of discontinuity. Notice that the single integral has no pole at $\omega = 0$ since $\mathrm{sinc}(0) = 1$, so there is no issue of principle part integrals involved. The poles only appeared above when we split the integral into two integrals.

10. The Area Rules and Parseval's Formulas

Setting $\omega = 0$ in the Fourier Transform (1.1) and then $t = 0$ in (1.2), one gets

$$X(0) = \int_{-\infty}^{\infty} dt\, x(t) = [\text{area under } x(t)] \tag{10.1}$$

$$x(0) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega) = (1/2\pi)\, [\text{area under } X(\omega)] \ . \tag{10.2}$$

For the box pulse example above, we saw that $X(0) = (A\tau)$ from (9.2). In light of (10.1), it is thus not a coincidence that this is the area under the time-domain box.

From (10.2), we may conclude that the total area under the $X(\omega)$ curve (9.2) for our box pulse is $2\pi A$, since $x(0) = A$, the height of our pulse. This is consistent with the fact that

$$\int_{-\infty}^{\infty} dx\, \text{sinc}(x) = \pi \ . \tag{10.3}$$

Another area rule involves the power spectrum. First, it is easy using (1.1), (1.2) and (2.1) to prove this identity (one of Parseval's),

$$\int_{-\infty}^{\infty} dt\, a(t)\, b^*(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, A(\omega)\, B^*(\omega) \ . \tag{10.4}$$

Here * means complex conjugation and is needed to make things work so one gets $\delta(\omega - \omega')$ in the proof. Again, the 2π factor is missing if one uses df in place of $d\omega$.

In the case $a = b = x$, one gets the energy area rule which says

$$\int_{-\infty}^{\infty} dt\, |x(t)|^2 = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, |X(\omega)|^2 = \int_{-\infty}^{\infty} df\, |\mathcal{X}(f)|^2 \ . \tag{10.5}$$

If $x(t)$ is a voltage or current pulse, this says that the total energy ($R = 1\Omega$) contained in the pulse is the same no matter which space is used to add it up. The pulse energy density is $|x(t)|^2$ in the time domain, it is $|X(\omega)|^2/2\pi$ in the ω domain, and it is $|\mathcal{X}(f)|^2$ in the frequency domain.

For our box pulse, the left side of (10.5) is $A^2\tau$. The right hand side gives the same result using (9.2) and the following fact,

$$\int_{-\infty}^{\infty} dx\, \text{sinc}^2(x) = \pi \ . \tag{10.6}$$

It is rather interesting that $\text{sinc}(x)$ and $\text{sinc}^2(x)$ have the exact same area, see (10.3) and (10.6).

```
plot([int(sin(x)/x,x),int((sin(x)/x)^2,x)], x = 0..50, color = [red,blue]);
```

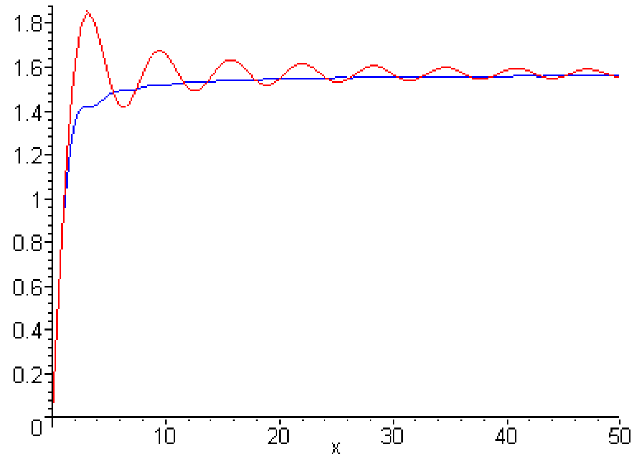


Fig 10.1

There are two other less-well-known Parseval's formulas which we just mention in passing,

$$\int_{-\infty}^{\infty} dt\ a(t)\ b(t) = (1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ A(\omega)\ B(-\omega) \tag{10.7}$$

$$\int_{-\infty}^{\infty} dt\ A(t)\ b(t) = \int_{-\infty}^{\infty} d\omega\ a(\omega)B(\omega)\ . \tag{10.8}$$

These appear in Stakgold Vol. 2, page 24 and elsewhere. All these formulas can be proven in the same manner: just use (1.1), (1.2) and (2.1).

11. Differentiation and Integration Rules with Examples

Below (4.8) and at the end of Section 8 we saw examples of how $d/dt \rightarrow i\omega$ in ω-space. Here we state the differentiation rule both ways:

$$dx(t)/dt \leftrightarrow [i\omega X(\omega)] \tag{11.1}$$

$$dX(\omega)/d\omega \leftrightarrow\ [-\ itx(t)\ ] \tag{11.2}$$

To derive the first rule in the general case we use (1.2) to expand x(t) so that

$$dx(t)/dt = d/dt\ [(1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ X(\omega)\ e^{+i\omega t}] = (1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ X(\omega)\ d/dt\ (e^{+i\omega t})$$

$$=\ (1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ X(\omega)i\omega\ (e^{+i\omega t}) = (1/2\pi)\ \int_{-\infty}^{\infty} d\omega\ [\ i\omega\ X(\omega)]\ e^{+i\omega t}\ .$$

Equation (11.2) has a similar derivation with a minus sign due to the sign of the exponent in (1.1).

So, (11.1) says that one gets the spectrum of the derivative of a function by multiplying the original function's spectrum by iω. For integration, one must therefore divide by iω.

(a) Let's apply (11.1) to our square pulse function. We have from (9.1) and (9.2),

$$x(t) = A\,[\,\theta(t + \tau/2) - \theta(t - \tau/2)\,]$$

$$X(\omega) = (A\tau)\,\mathrm{sinc}(\omega\tau/2)\ .$$

Differentiating x(t), we get a pair of opposite signed delta functions separated by distance τ (derivatives of the box edges),

$$x_1(t) \equiv dx(t)/dt = A\,[\,\delta(t + \tau/2) - \delta(t - \tau/2)\,]\ .$$

According to (11.1), the spectrum must be,

$$X_1(\omega) = i\omega\,(A\tau)\,\mathrm{sinc}(\omega\tau/2) = i\omega\,(A\tau)\sin(\omega\tau/2)/\,(\omega\tau/2) = 2iA\,\sin(\omega\tau/2)\ .$$

This agrees with direct calculation,

$$X_1(\omega) = \int_{-\infty}^{\infty} dt\; x_1(t)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt\; A\,[\,\delta(t + \tau/2) - \delta(t - \tau/2)\,]\, e^{-i\omega t}$$

$$= A\,[e^{i\omega\tau/2} - e^{-i\omega\tau/2}] = 2iA\,\sin(\omega\tau/2)\ .$$

As expected, there is no DC component since $\lim_{\omega\to 0} X_1(\omega) = 0$. In this drawing,

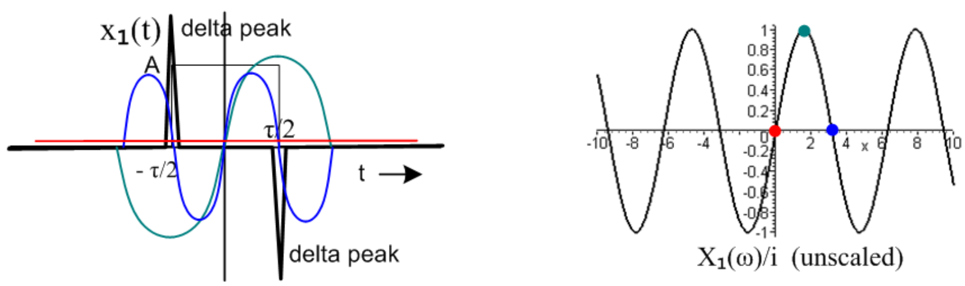


Fig 11.1

we see $x_1(t)$ on the left in heavy black, and the spectrum $X_1(\omega)$ is on the right. The red line and dot show that there is zero energy at DC, ω = 0. The green dot on the right at the first peak of $X_1(\omega)/i$ corresponds to the green sine curve on the left, which we would expect to give a strong component for the double delta. To understand the sign of the green dot, recall that (1.9) for an odd function $x_1(t)$ states $X_1(\omega) = -\ i\ X_{1,s}(\omega)$ so that $X_1(\omega)/i = - X_{1,s}(\omega)$ = the negative of the Fourier Sine projection.

(b) Now let's apply (11.2) to the following function (multiply box x(t) above by t)

$$x_2(t) \equiv t\,x(t) = A\,t\,[\,\theta(t + \tau/2) - \theta(t - \tau/2)\,].$$

This represents a doublet sawtooth pulse centered at t=0. According to (11.2) in the ← direction,

$$X_2(\omega) = i\, dX(\omega)/d\omega \;=\; i\,(A\tau)(\tau/2)\,\mathrm{sinc}'(\omega\tau/2) \;=\; (\,iA\tau^2/2)\,\mathrm{sinc}'(\omega\tau/2)$$

where $\mathrm{sinc}'(x) = \cos(x)/x - \sin(x)/x^2$. Again, $X_2(\omega)$ has no DC component, since $\lim_{x\to 0}\mathrm{sinc}'(x) = 1/x - 1/x = 0$. This is an agreement with the fact that the sawtooth clearly has a zero integral and this integral according to (1.1) is just X(0). Here is a picture similar to that shown above,

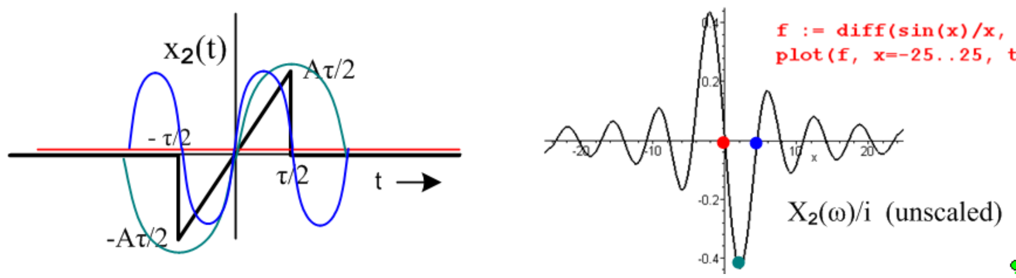


Fig 11.2

12. Time translation x(t) causes phase on X(ω).

Assume that some x(t) has a spectrum X(ω),

$$x(t) \quad\leftrightarrow\quad X(\omega)$$

Then it follows directly from (1.2) that:

$$x(t - t_1) \quad\leftrightarrow\quad X(\omega)\, e^{-i\omega t_1} \quad . \qquad (12.1)$$

We saw this happening in the special case of (8.2); here we see that the result is completely general. Translation of a signal in time causes the spectrum to gain the phase shown.

According to (1.1), we have this analogous result ,

$$X(\omega-\omega_1) \quad\leftrightarrow\quad x(t)\, e^{+i\omega_1 t} \qquad (12.2)$$

13. Exponential Sum Rules

For the first time in this document, sums appear. Everything above was integrals only.

In (2.1) stated above, the delta function is written as an infinite integral of an exponential,

$$\int_{-\infty}^{\infty} dx\, e^{\pm ikx} = 2\pi\delta(k) \,. \qquad (2.1)$$

A similar result involves a summation of exponentials. For k in the range -π to π we claim that:

$$\sum_{n=-\infty}^{\infty} e^{ink} = 2\pi\delta(k) \qquad -\pi < k < \pi \;. \qquad (13.1)$$

To "prove" this, we argue as we did for (2.1) that for k ≠ 0, the phasors "wash out" in the infinite sum and we get zero = zero. For k = 0, the summand is 1, so the result is infinite, and thus the result is proportional to δ(k). As we did above, we can prove that the factor of 2π is correct by integrating both sides over k from -a to +a. The right side gives 2π. On the left, do the dk integral exactly as done above (2.2) to get

$$\int_{-a}^{a} dk \sum_{n=-\infty}^{\infty} e^{ink} = \sum_{n=-\infty}^{\infty} \int_{-a}^{a} dk\, e^{ink} = \sum_{n=-\infty}^{\infty} \int_{-a}^{a} dk \cos(nk) \qquad // \sin(nk) \text{ is odd}$$

$$= \sum_{n=-\infty}^{\infty} [2\sin(na)/n\,] = 2a + 4\left\{ \sum_{n=1}^{\infty} [\sin(na)/n\,] \right\} = 2a + 4\left\{ \frac{\pi - a}{2} \right\} = 2\pi \,.$$

In the last few steps, the negative part of the sum is reflected into a positive part since [2sin(na)/n] is even in n. The sum in curly brackets appears as 1.441.1 on p 46 of Gradshteyn-Ryzhik,

1.441

1. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad [0 < x < 2\pi] \qquad \text{FI III 539}$$

This sum is restricted then to 0 < a < 2π, but since we had in mind a being some small positive number, this is not a problem, although it does provide a hint of what is to come below.

Since the right side of (13.1) is real, the equation is also valid with e^{-ink} on the left.

Now we want to generalize (13.1) for k in the range -∞ to +∞. The result is fairly obvious. Instead of just a delta function at k=0, we have delta function spikes at each k value for which the summand equals 1. Thus, spikes will be at k = 0, ± 2π, ± 4π, and so on,

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \;. \qquad (13.2)$$

Again, one can prove that the constant is 2π at each spike by integrating over dk from 2πm-a to 2πm+a, for small a. Generally speaking, the right side of (13.2) must have the form shown because the left side is periodic in k with period 2π, and (13.1) gives that result for -π < k < π .

Replacing n→-n shows that (13.2) is also valid with e^{-ink} on the left side.

The exponential sum rule (13.2) plays a critical role in the analysis of periodic pulse trains in Chapter 2 below.

Equations (2.1) and (13.2) are more carefully derived in Appendix A. There it is shown that in each case the delta function is the limit of a certain sequence of functions which all have unit area and which, as the limit is taken, become more and more isolated to the neighborhood of the delta function argument. The Appendix presents the essence of the distribution theory of delta functions.

One result derived in Appendix A (b) is a finite sum version of (13.2) [see (A.29) and (A.30)]

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \quad . \qquad -\infty < k < \infty \qquad (13.3)$$

In the limit $N\to\infty$, Appendix A shows how the right side of (13.3) approaches the right side of (13.2).
Replacing $n\to -n$ shows that (13.3) is also valid with e^{-ink} on the left side.

<u>Poisson Sum Formula</u>: Setting $k = 2\pi t/\alpha$ in (13.2) with e^{-ink} gives (where α is any real number)

$$\sum_{n=-\infty}^{\infty} e^{-in2\pi t/\alpha} = \sum_{m=-\infty}^{\infty} \delta(t/\alpha - m) = |\,\alpha\,| \sum_{m=-\infty}^{\infty} \delta(t - m\alpha) \ . \qquad (13.4)$$

Applying $\int_{-\infty}^{\infty} dt\, x(t)$ to both sides and using (1.1) one finds that

$$\sum_{n=-\infty}^{\infty} X(2\pi n/\alpha) = |\,\alpha\,| \sum_{m=-\infty}^{\infty} x(m\alpha) \ . \qquad (13.5)$$

This fascinating result appears for example in Stakgold Vol I (1.23b) and is known as the Poisson Sum Formula. Sometimes people refer to the underlying equation (13.2) by this name.

Chapter 2: Pulse Trains and the Fourier Series Connection

In this chapter we take an arbitrarily shaped pulse and superpose an infinite number of instances of that pulse spaced by a fixed time interval T_1. This "pulse train" is then a periodic function of period T_1. We continue with the Fourier Integral notions of Chapter 1 -- such as the spectral components $X(\omega)$ -- and we then make the connection with traditional Fourier Series and their coefficients. We show how the Fourier Integral spectrum becomes discrete for a periodic function, and we are able then to relate the Fourier Series coefficients to the spectral components $X_{pulse}(\omega)$ of the pulse used to generate the pulse train.

In the background, and to serve as a vehicle for doing a few calculations, we address a particular problem. We consider the symmetric zero-DC-offset square-wave pulse train generated from two completely different methods, one involving adding a negative DC offset to a simple positive square wave pulse train, and the other using biphase pulses.

Our main purpose here is to build tools that will be used in more complicated problems. The methods presented here form the basis for treating amplitude-modulated pulse trains made from pulses of arbitrary shape, including the "arbitrariness" of a pulse being statistically present or absent.

14. The Spectrum of a Simple Pulse Train

Recall that a simple pulse train is just a sequence of identical pulses of shape $x_{pulse}(t)$.

(a) Infinite Length Simple Pulse Train

We assume that $x_{pulse}(t)$ is some "reasonable" (non-pathological) function. The pulse train is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - t_n) \qquad t_n = nT_1 . \qquad \text{pulse train} \qquad (14.1)$$

Let $X_{pulse}(\omega)$ be the spectrum of $x_{pulse}(t)$. This $x_{pulse}(t)$ does not really have to be a "pulse", but it is convenient to think of it as such. We imagine that $x_{pulse}(t)$ is a function that is somewhat localized in the region of t=0, and vanishes for very large positive and negative time. The half-width of the pulse can be larger than T_1 as discussed below, so pulses can overlap. Thus, our $x_{pulse}(t)$ is not itself periodic, and we thus expect *it* to have a continuous spectrum.

For example, from (9.2) we already know the spectrum of a single box pulse of height A and width τ centered at t=0:

$$X_{pulse}(\omega) = (A\tau)\, \text{sinc}(\omega\tau/2) . \qquad (14.2)$$

Now consider a second pulse which is a copy of our original pulse, but which is translated T_1 units to the right in time. From (12.1), we know the spectrum of this second pulse:

$$X(\omega, \text{second pulse}) = X_{pulse}(\omega)\, e^{-i\omega T_1} .$$

Now construct an infinite periodic wave by superposing pulses at t = 0, $\pm T_1$, $\pm 2T_1$, ... We get,

$$X(\omega) = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} e^{i\omega n T_1} \tag{14.3}$$

According to our exponential sum rule (13.2) with $k = \omega T_1$, we can write the exponential sum in (14.3) as a sum of delta functions to get,

$$X(\omega) = X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) . \tag{14.4}$$

Defining $\omega_1 \equiv 2\pi/T_1$ this becomes

$$X(\omega) = (1/T_1)\, X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - m\omega_1) . \tag{14.5}$$

which is a standard form. Moving $X_{pulse}(\omega)$ into the sum then gives

$$X(\omega) = \sum_{m=-\infty}^{\infty} (1/T_1)\, X_{pulse}(\omega)\, 2\pi\delta(\omega - m\omega_1)$$

$$= \sum_{m=-\infty}^{\infty} (1/T_1)\, X_{pulse}(m\omega_1)\, 2\pi\delta(\omega - m\omega_1) . \tag{14.6}$$

Thus, we have a set of evenly spaced delta function spikes which occur at these frequencies:

$$\omega_m = m\omega_1 \quad m = 0,1,2,3...... \tag{14.7}$$

In general, one has,

$$f(\omega)\, \delta(\omega - a) = f(a)\, \delta(\omega - a) .$$

Both sides of this last equation are zero when $\omega \neq a$, and at $\omega = a$, $f(a) = f(\omega)$.

Because the quantity $(1/T_1)X_{pulse}(\omega)$ occurs frequently in the following discussion, we define a more compact notation for it as follows:

$$c(\omega) \equiv (1/T_1)X_{pulse}(\omega) . \tag{14.8}$$

Thus, $c(\omega)$ is nothing more than our (continuous) pulse spectrum divided by the fundamental period T_1. We can then rewrite (14.6) as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\ \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c(\omega_m)\ 2\pi\ \delta(\omega - m\omega_1)\ . \qquad (14.9)$$

As was just noted above, we can harmlessly replace ω with $\omega_m = m\omega_1$ inside $c(\omega)$ in (14.8). This leads us to define a set of numbers as follows

$$c_m \equiv c(\omega_m) = c(m\omega_1)\ . \qquad (14.10)$$

These numbers are just the values that the function $c(\omega)$ takes at our delta spike frequencies. We arrive then at our final form for the spectrum of an infinite simple pulse train,

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1)\ . \qquad (14.11)$$

Now we are ready to summarize all these results:

Fourier Integral Transform of an Infinite Simple Pulse Train (14.12)

1. Let $x_{pulse}(t)$ be any reasonable pulse. Construct a pulse train $x(t)$ with spacing T_1:

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1)\ . \qquad \textit{(14.1)}$$

By its construction, $x(t)$ is periodic with period T_1, which we can write formally as:

$$x(t + nT_1) = x(t)\ . \qquad n = \text{any integer}$$

If $x(t)$ is a *known* periodic function of period T_1, a candidate for $x_{pulse}(t)$ is $x(t)$ over any one period.

2. Define $c(\omega)$ to be the Fourier Integral transform of the pulse, scaled by $1/T_1$:

$$c(\omega) \equiv (1/T_1)X_{pulse}(\omega) = (1/T_1) \int_{-\infty}^{\infty} dt\ x_{pulse}(t)\ e^{-i\omega t}\ . \qquad \textit{(14.8) and (1.1)}$$

3. Then the Fourier Integral transform of the Pulse *Train* is as follows:

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\ \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1) \qquad \textit{(14.9)}$$

where $c_m = c(\omega_m)$, $\omega_m = m\omega_1$, $\omega_1 = 2\pi/T_1$.

4. These c_m are the same c_m which appear in the next section. That is, they the complex Fourier Series coefficients.

Item 3 is our main result. It says that the Fourier Transform spectrum of an infinite sequence of pulses is a sum of equally-spaced delta function spikes whose coefficients are given by the continuous spectrum of the central pulse evaluated at the spike frequencies $\omega = m\omega_1$. The pulse spectrum $c(\omega) = (1/T_1)X_{pulse}(\omega)$ is normally thought of as the "coefficient envelope", while the equally spaced delta function spikes are the "lines". In this sample symbolic drawing of a spectrum, the infinitely-high delta function spikes of the spectrum are represented by finite vertical red line segments whose heights are the coefficients c_m. The red lines *are* the spectrum, and they track the envelope $c(\omega)$.

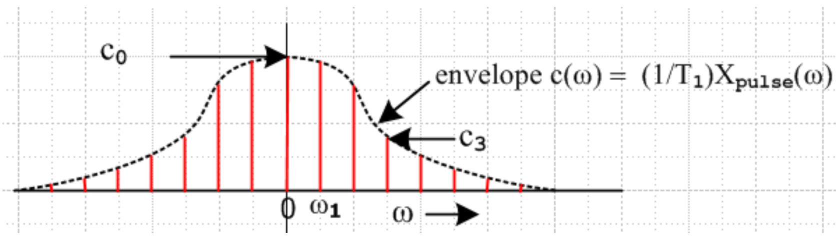


Fig 14.1

It may happen that certain c_m vanish, meaning that such lines are not present.

The item 3 sum includes the DC line m=0 having $\omega_0 = 0$. Unless c_0 happens to vanish, the pulse train has a DC component. As noted in (10.1), $X_{pulse}(0)$ is the area under $x_{pulse}(t)$. Only if this area is zero do we get $c_0 = c(0) = (1/T_1)\, X_{pulse}(0) = 0$.

Whereas the Fourier Transform spectrum of a single pulse (localized, non-periodic) is continuous in ω, that of an infinite sequence of pulses is entirely discrete and has no continuous portions. This conforms with the well-known fact that the spectrum of any periodic function is discrete. In fact, we have just proven this to be so. Any periodic signal has to repeat some pattern, and we just take that pattern to be our $x_{pulse}(t)$.

Note on $x_{pulse}(t)$

In our summary box (14.12) above, we say that if x(t) is some *known* periodic function, one can take as a *candidate* for $x_{pulse}(t)$ the function x(t) restricted to any one period. In this case, the dt integration endpoints for the projection $X_{pulse}(\omega)$ only cover that selected period. If we select the period centered at t=0, then item 2 in the above summary box becomes perhaps more familiar:

$$c(\omega) = (1/T_1)X_{pulse}(\omega) = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-i\omega t} \quad . \qquad (14.13)$$

What is perhaps less obvious is that there are many different candidates for $x_{pulse}(t)$ that result in the *same* x(t) pulse train. These other choices for $x_{pulse}(t)$ are pulses which slop over into more than one period T_1. When a pulse train is formed with such pulses, the pulses overlap.

To see how this might work, think of a pulse which has a nice gaussian shape and goes about half way into each neighboring T_1 interval. Draw some of these, then add them up to make the sum curve x(t). In this case, for a candidate $x_{pulse}(t)$, one can use either the gaussian, which overlaps into several intervals, or one can use one interval's worth of the sum curve x(t) (shown as the darker curve)

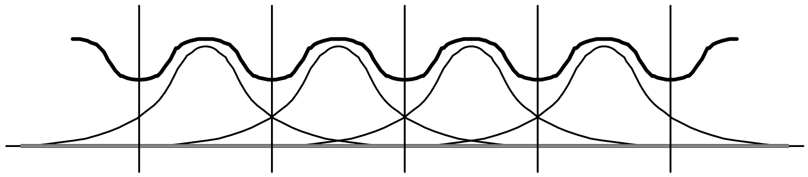


Fig 14.2

We have tried to keep our formulas completely general to allow for pulse trains formed from pulses which overlap into more than one period. The resulting spectrum is of course the same no matter which $x_{pulse}(t)$ is chosen. Later on in the power discussion we shall always regard $x_{pulse}(t)$ as meaning the shape of x(t) over T_1, as indicated by the black curve above.

To summarize, we can write c(ω) in two equivalent ways

$$c(\omega) = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-i\omega t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-i\omega t} \quad . \qquad (14.14)$$

If we evaluate (14.14) at the discrete spike frequencies $\omega = m\omega_1$, we get

$$c_m = (1/T_1) \int_{-T_1/2}^{T_1/2} dt\, x(t)\, e^{-im\omega_1 t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-im\omega_1 t} \quad . \qquad (14.15)$$

Now since $e^{-im\omega_1(t+T_1)} = e^{-im\omega_1 t}\, e^{-im\omega_1 T_1} = e^{-im\omega_1 t}\, e^{-im2\pi} = e^{-im\omega_1 t}$, function $e^{-im\omega_1 t}$ is periodic with period T_1. Since x(t) in the first integral in (14.15) is also assumed periodic with period T_1, the integration can be over any interval of width T_1, so one usually takes this interval to be $(0,T_1)$. Thus,

$$c_m = (1/T_1) \int_0^{T_1} dt\, x(t)\, e^{-im\omega_1 t} = (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-im\omega_1 t} \qquad (14.16)$$

(b) Finite Length Simple Pulse Train

It is a simple matter to modify the above development for a finite length pulse train. We use a finite pulse train having pulses centered about time t = 0 as in (14.17), then (14.3) becomes (14.18) below :

$$x(t) = \sum_{n=-N}^{N} x_{pulse}(t - t_n) \qquad t_n = nT_1 \, , \qquad \text{pulse train} \qquad (14.17)$$

$$X(\omega) = X_{pulse}(\omega) \sum_{n=-N}^{N} e^{i\omega n T_1} \quad . \qquad (14.18)$$

The sum is done using (13.3) to give

$$X(\omega) = X_{pulse}(\omega)\, 2\pi\delta_5(\omega T_1,N) \qquad = c(\omega)\, 2\pi\, T_1\delta_5(\omega T_1,N) \tag{14.19}$$

where δ_5 is a delta function model described in Appendix A. This δ_5 is periodic with period 2π and has identical peaks separated by 2π. For large N we know that

$$\delta_5(k,N) \approx \sum_{m=-\infty}^{\infty} \delta_4(k - 2\pi m, N+1/2) \qquad \text{for large N} \tag{A.18}$$

where $\delta_4(x,M)$ is another delta function model which peaks only near x=0. Thus for large N we can write

$$X(\omega) \approx c(\omega) \sum_{m=-\infty}^{\infty} 2\pi\, T_1\delta_4(\omega T_1 - 2\pi m, N+1/2)\ . \tag{14.20}$$

To the extent that these δ_4 peaks are very narrow (large N) we can move $c(\omega)$ inside the sum and evaluate it at $\omega T_1 = 2\pi m$ (which means $\omega = m\omega_1$) to get

$$X(\omega) \approx \sum_{m=-\infty}^{\infty} c_m\ 2\pi\, T_1\delta_4(\omega T_1 - 2\pi m, N+1/2)\ . \tag{14.21}$$

In the limit $N\rightarrow \infty$, we get $\delta_4(\omega T_1 - 2\pi m, N+1/2) \rightarrow \delta(\omega T_1 - 2\pi m)$ and then

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\, T_1\, \delta(\omega T_1 - 2\pi m) \quad = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\, \delta(\omega - m\omega_1)$$

which agrees with (14.11).

<u>Example:</u> Suppose the pulse is our usual box of width T_1 and height A. Then the pulse train is a constant DC level x(t) = A. We know that all the c_m will vanish except c_0 which is easy to evaluate

$$c_0 = (1/T_1) \int_{-\infty}^{\infty} dt\ x_{pulse}(t) = (1/T_1)\,(AT_1) = A\ .$$

The spectrum is then

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1) = c_0 2\pi\delta(\omega) = A\ 2\pi\delta(\omega)$$

which is a delta line at $\omega = 0$ with factor $2\pi A$, consistent with (8.1).

15. Connection with the traditional Fourier Series

The pulse train spectrum (14.11) may be inserted into the Fourier transform expansion (1.2) to get

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega \left[\sum_{m=-\infty}^{\infty} c_m\, 2\pi\, \delta(\omega - m\omega_1) \right] e^{+i\omega t}$$

$$= \sum_{m=-\infty}^{\infty} c_m \int_{-\infty}^{\infty} d\omega\, \delta(\omega - m\omega_1)\, e^{+i\omega t} = \sum_{m=-\infty}^{\infty} c_m\, e^{+im\omega_1 t} \qquad (15.1)$$

$$= c_0 + \sum_{m=1}^{\infty} [\, c_m e^{+im\omega_1 t} + c_{-m} e^{-im\omega_1 t}] = c_0 + \sum_{m=1}^{\infty} [\, c_m e^{+im\omega_1 t} + (c_m e^{+im\omega_1 t})^* \,]$$

$$= c_0 + \sum_{m=1}^{\infty} 2\, \mathrm{Re}\, [c_m e^{+im\omega_1 t}] = c_0 + 2\, \mathrm{Re} \left[\sum_{m=1}^{\infty} c_m e^{+im\omega_1 t} \right] \quad . \qquad (15.2)$$

In the above we have used the reflection rule (7.3) applied to $c_m \equiv (1/T_1) X_{pulse}(\omega_m)$ to find that $c_{-m} = c_m^*$, and then $c_{-m} e^{-im\omega_1 t} = (c_m e^{+im\omega_1 t})^*$.

We know that the c_m are in general complex numbers, so make the following two definitions:

$$a_m \equiv 2\, \mathrm{Re}\, [\, c_m \,] = (2/T_1)\, \mathrm{Re}\, [\, X_{pulse}(m\omega_1) \,]$$

$$- b_m \equiv 2\, \mathrm{Im}\, [\, c_m \,] = (2/T_1)\, \mathrm{Im}\, [\, X_{pulse}(m\omega_1) \,] \quad . \qquad (15.3)$$

Since $c_m = (1/T_1) X_{pulse}(m\omega_1)$ it follows that

$$c_m = (1/2)\, [\, a_m - i b_m \,] \, . \qquad (15.4)$$

From (14.16) , assuming as we do from now on that x(t) is real, we get

$$c_m = (1/T_1) \int_0^{T_1} dt\, x(t)\, e^{-im\omega_1 t} \qquad (15.5)$$

$$= (1/T_1) \int_0^{T_1} dt\, x(t)\, [\, \cos(m\omega_1 t) - i \sin(m\omega_1 t)]$$

so that

$$a_m = 2\, \mathrm{Re}\, [\, c_m \,] = (2/T_1) \int_0^{T_1} dt\, x(t) \cos(m\omega_1 t) \qquad (15.6)$$

$$b_m = -2\, \mathrm{Im}\, [c_m \,] = (2/T_1) \int_0^{T_1} dt\, x(t) \sin(m\omega_1 t) \quad . \qquad (15.7)$$

In all the above integrals, we can replace $\int_0^{T_1} dt\, x(t)$ by $\int_{-\infty}^{\infty} dt\, x_{pulse}(t)$ as noted in (14.16).

Since x(t) is real, we know from (7.3) that X(-ω) = X(ω)*, so X(0) must be real. We also know this from the "area rule" (10.1) -- the area under a real function x(t) had better be real. Thus from (15.3) $b_0 = 0$ and from (15.4) $c_0 = a_0/2$ so that

$$\text{DC component of } x(t) = c_0 = (a_0/2) = (1/T_1)\ X_{pulse}(0)\ . \tag{15.8}$$

If we install expression (15.4) for c_m into (15.2) we get this result:

$$x(t) = c_0 + 2\,\mathrm{Re}\,[\ \sum_{m=1}^{\infty} c_m\, e^{+im\omega_1 t}\ ] \;=\; a_0/2 \;+\; \sum_{m=1}^{\infty} \mathrm{Re}\ \{\ [\ a_m - ib_m\ ]\ [\ \cos(m\omega_1 t) + i\,\sin(m\omega_1 t)]\ \}$$

$$= a_0/2 \;+\; \sum_{m=1}^{\infty} a_m \cos(m\omega_1 t) + \sum_{m=1}^{\infty} b_m \sin(m\omega_1 t) \qquad\qquad \omega_1 = 2\pi/T_1 \tag{15.9}$$

This expansion, along with projections (15.6) and (15.7), is the traditional Fourier Series expansion of a periodic function of period T_1. Thus, our seemingly uninteresting a_m and b_m coefficients are exactly the standard Fourier Series coefficients. Moreover, the DC component of x(t) is equal to $c_0 = (a_0/2)$.

For completeness, we write down an alternate form of (15.9),

$$x(t) = a_0/2 \;+\; \sum_{m=1}^{\infty} A_m \cos(m\omega_1 t + \varphi_m) \tag{15.10}$$

$$= a_0/2 \;+\; \sum_{m=1}^{\infty} A_m\,[\cos(m\omega_1 t)\cos(\varphi_m) - \sin(m\omega_1 t)\sin(\varphi_m)\ ]$$

$$= a_0/2 \;+\; \sum_{m=1}^{\infty} [A_m \cos(\varphi_m)]\cos(m\omega_1 t) \;+\; \sum_{m=1}^{\infty} [-A_m \sin(\varphi_m)]\sin(m\omega_1 t)$$

Thus,

$$a_m = A_m \cos(\varphi_m) \qquad\qquad A_m = \sqrt{a_m^2 + b_m^2}$$

$$-b_m = A_m\ \sin(\varphi_m) \qquad\qquad \tan(\varphi_m) = -b_m/a_m\ . \tag{15.11}$$

So, here is a summary of the above efforts:

Fourier Series Transform (15.12)

1. Let $x_{pulse}(t)$ be any reasonable pulse. Construct a pulse train x(t) with spacing T_1:

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) \qquad (14.1)$$

By its construction, x(t) is periodic with period T_1, which we can write formally as:

$$x(t + nT_1) = x(t) \qquad n = \text{any integer}$$

If x(t) is a *known* periodic function of period T_1, a candidate for $x_{pulse}(t)$ is x(t) over any one period.

2. Define the Fourier Series coefficients by these projections = transforms ($c_m = [\, a_m - ib_m \,]/2$)

$$c_m \equiv (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-im\omega_1 t} = (1/T_1) \int_0^{T_1} dt\, x(t)\, e^{-im\omega_1 t} \qquad (14.16)$$

$$a_m \equiv (2/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t) \cos(m\omega_1 t) = (2/T_1) \int_0^{T_1} dt\, x(t) \cos(m\omega_1 t) \qquad (15.6)$$

$$b_m \equiv (2/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t) \sin(m\omega_1 t) = (2/T_1) \int_0^{T_1} dt\, x(t) \sin(m\omega_1 t) \qquad (15.7)$$

3. The pulse train is then given by these expansions = inverse transforms: ($\omega_1 = 2\pi/T_1$)

$$x(t) = \sum_{m=-\infty}^{\infty} c_m e^{+im\omega_1 t} = a_0/2 + \sum_{m=1}^{\infty} a_m \cos(m\omega_1 t) + \sum_{m=1}^{\infty} b_m \sin(m\omega_1 t) \qquad (15.9)$$

Note that a_m, b_m, c_m and x(t) all have the same dimensions, perhaps volts.

Thus, we have derived the Fourier Series Transform from the Fourier Integral Transform. Recall that it is not necessary that $x_{pulse}(t)$ be totally contained within a width T_1 We have infinite endpoints on the dt integrations above, and $x_{pulse}(t)$ is allowed to be any "reasonable" function, meaning the projection integrals must converge.

16. Fourier Series for a positive square wave pulse train

From equation (9.2) the Fourier Integral spectrum of a positive box pulse of width τ and height A is

$$X_{pulse}(\omega) = (A\tau)\ \mathrm{sinc}(\omega\tau/2)\ . \qquad (16.1)$$

From (14.8) and (14.10) the complex Fourier Series coefficients are, using $\omega_1 = 2\pi/T_1$,

$$c_m = (1/T_1)\ X_{pulse}(m\omega_1) = (A\tau/T_1)\ \mathrm{sinc}(m\pi\tau/T_1)\ . \qquad (16.2)$$

Thus, from (15.3), we know the a and b coefficients as well:

$$a_m = (2A\tau/T_1)\ \mathrm{sinc}(m\pi\tau/T_1) \qquad m = 0,1,2,3... \qquad (16.3)$$
$$b_m = 0. \qquad m = 0,1,2,3...$$

We have here the Fourier Series coefficients for an infinite pulse train of positive pulses of amplitude A, width τ, and period T_1, such that the time t=0 occurs in the middle of a positive pulse. If $T_1 = \tau$, the pulse train is a constant DC level A and $c_m = \delta_{m,0}A$, a case of minimal interest, so we assume $T_1 > \tau$:

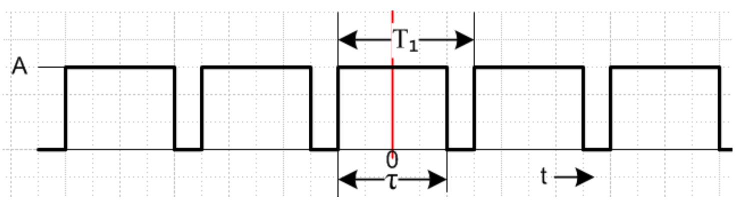


Fig 16.1

The reader is invited to compute the above Fourier Series coefficients in the standard manner, using the conventional formulas in the summary box (15.12). This is done also on page 32-33 of Bennett and Davey (who use $T_1 = T$). Their result agrees with the above.

17. More about positive square-wave pulse trains

We can now summarize what we know about the positive square-wave pulse train with pulse height A, pulse width τ, pulse centered at t = 0, and period T_1 (with $\omega_1 = 2\pi/T_1$, see drawing above) :

$$x_{pulse}(t) = A\ [\ \theta(t + \tau/2) - \theta(t - \tau/2)\ ]. \qquad \textit{(9.1)} \qquad (17.1)$$

$$c(\omega) = (1/T_1)\ X_{pulse}(\omega) = (A\tau/T_1)\ \mathrm{sinc}(\omega\tau/2) \qquad \textit{(9.2) and (14.8)} \qquad (17.2)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) \qquad \textit{(14.12)} \qquad (17.3)$$

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\delta(\omega - m\omega_1) \qquad \textit{(14.11)} \qquad (17.4)$$

$$c_m = (1/T_1)\ X_{pulse}(m\omega_1) = (A\tau/T_1)\ \mathrm{sinc}(m\pi\tau/T_1) \qquad \textit{(17.2)}\ \omega = m\omega_1 \qquad (17.5)$$

$$c_0 = (A\tau/T_1) = \text{DC component}\ . \qquad \textit{(15.8) and (14.2)} \qquad (17.6)$$

For general τ, all spectral lines are present. Apart from an overall constant, the envelope function $c(\omega)$ is sinc(x), where $x = \omega\tau/2$. Here is a linear graph of |sinc(x)| = |sin(x)/x| (red) along with a graph of 1/x (blue). It is traditional to plot the absolute value of the spectrum since the power in each line is proportional to the square $|c_m|^2$ (shown later in (33.27)),

```
plot([abs(sin(x)/x), 1/x], x = 0..15, y = 0..1,thickness=2, color=[red,blue],
tickmarks = [10,10]);
```

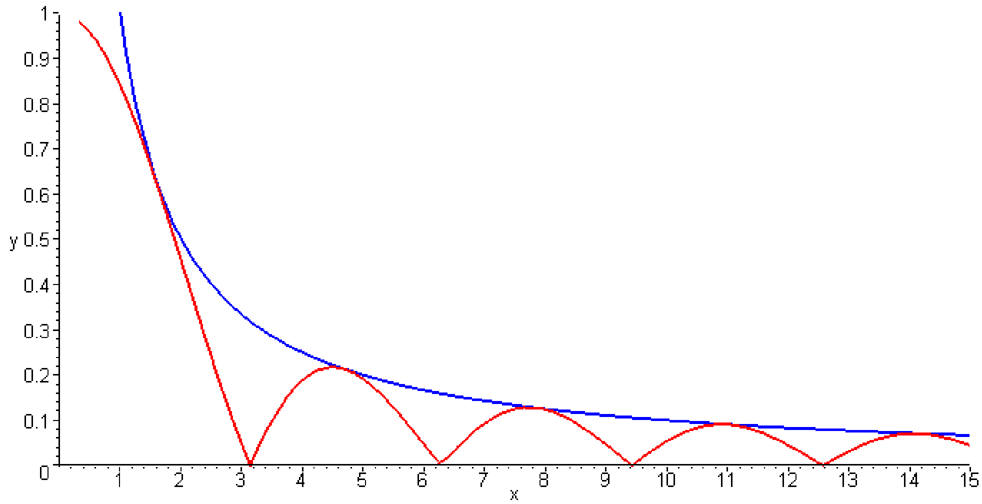


Figure 17.1: Plot of |sinc(x)| function along with 1/x. Fig 17.1

In the case of general pulse width τ (that is, for arbitrary pulse train **duty cycle** = τ/T_1), one should imagine the evenly-spaced delta spikes superposed on the above picture. The spikes are located at $x_m = \omega_m\tau/2 = m\omega_1\tau/2 = m(\pi\tau/T_1)$, for m = 1,2,3... The spacing between the spikes is $dx = (\tau/T_1)\pi$. Thus, the number of spikes per hump is (T_1/τ) since each hump is π wide. At low duty cycle, the spacing is small, and there are many lines for each hump of the |sinc(x)| curve. Here is a rough plot for an ~8% duty cycle, $(T_1/\tau) = 16$:

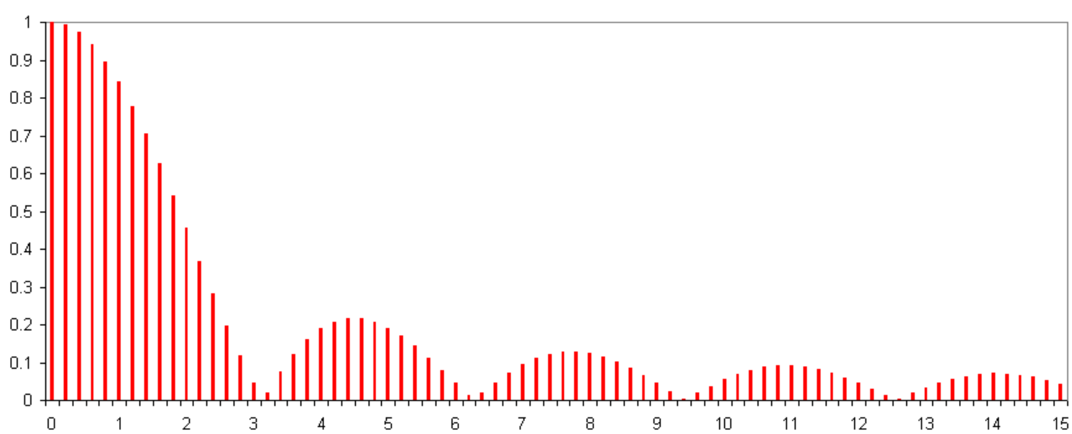


Figure 17.2. Same |sinc(x)| function with delta spike "lines". Height of each line is relative magnitude of the c_m coefficient. Plot is for $\tau = T_1/16$, duty cycle about 8% . Fig 17.2

We shall now examine some special cases as application of what has so far been established.

(a) If $\tau = T_1/2$ (50% duty cycle) we get a symmetric positive square wave pulse train,

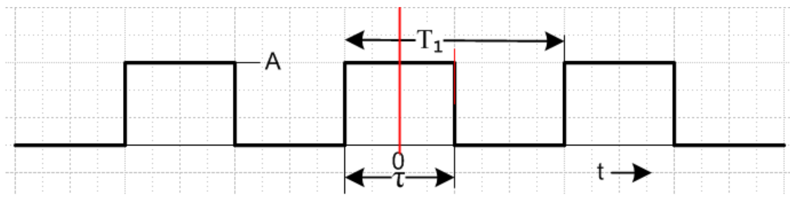


Fig 17.3

and (17.5) reduces to

$$c_m = (A/2)\ \mathrm{sinc}(m\pi/2). \qquad (17.7)$$

The DC component (the m=0 line) is $c_0 = (A/2)$, which is what we expect. All the other even lines vanish due to the sinc form. For m = odd integers, we know that

$$\sin(m\pi/2) = (-1)^{(1-m)/2} = (i)^{1-m} = \text{ real, since m odd} \qquad (17.8)$$

We summarize these facts for our symmetric pulse train with t=0 centered on a positive pulse:

$$c_m = (A/\pi)\,(i)^{1-m}\,(1/m) \qquad m = \text{odd} \qquad (17.9)$$
$$c_m = 0 \qquad m = \text{even},\ m \neq 0$$
$$c_0 = (A/2)$$

(If we were to shift the square wave down A/2, the c_m would be the same except $c_0 = 0$.)

For Fig 17.3, if A = 2 we get

$c_1 = (2/\pi) = 0.64$ $\qquad$ $c_3 = -(1/3)(2/\pi) = -0.21$ $\qquad$ $c_5 = (1/5)(2/\pi) = 0.13$.

In terms of Figure 17.1, the spacing between the spike positions is $\pi/2$. Thus, all the m=even spikes occur exactly at the zeros of the sinc(x) function, that is why they all vanish. The m=odd spikes occur centered between these zeros, very close to the peaks of the humps. The 1/m drop-off of the Fourier coefficients seen in (17.9) is reflected in our plot of 1/x in the picture. The 1/x curve intersects the odd spikes at the c_m coefficient values which are dropping off as 1/m (apart from overall constant). This plot uses A = 2 :

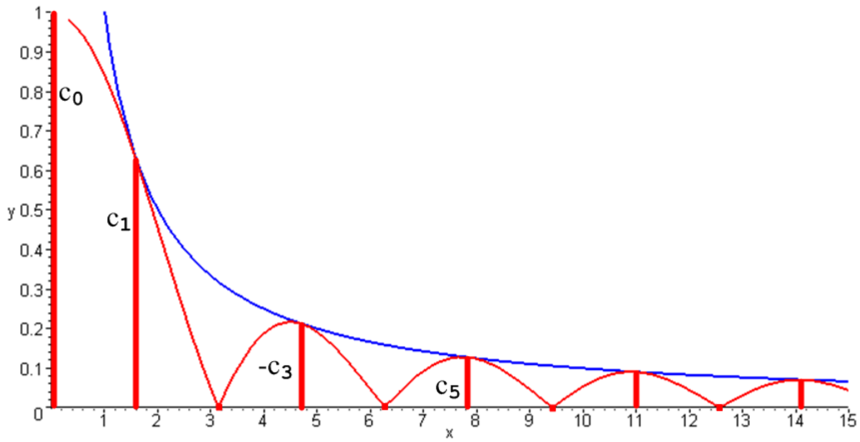


Fig 17.4

Since all c_m in (17.9) are real, we know that $b_n = 0$ so there are only Fourier Series cosine contributions to x(t). This is pretty clear looking at the time domain waveform shown in Fig 17.3 which is even in t.

The pulse train we have constructed above has t=0 occurring in the middle of a positive pulse. If we were to shift our entire pulse train to the left by τ/2,

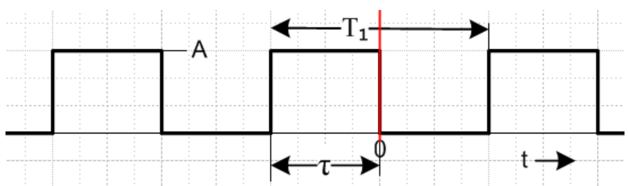


Fig 17.5

so that falling pulse edge lines up with t=0, we would acquire an overall factor $e^{i\omega\tau/2}$ according to (12.1) which should then be added as a factor to (17.4). At the lines $\omega = m\omega_1$ this factor becomes $e^{+im\omega_1\tau/2} = e^{+im(2\pi/T_1)(T_1/4)} = e^{im\pi/2} = (i)^m$ acting on the c_m coefficients. This cancels the phase shown in (17.9) leaving only a constant i.

So, here are the results for the same pulse train with t=0 occurring at a falling edge:

$$c_m = i\,(A/\pi)\,(1/m) \qquad m = \text{odd} \tag{17.10}$$
$$c_m = 0 \qquad m = \text{even},\ m \neq 0$$
$$c_0 = (A/2)$$

Since all the odd-m c_m are now imaginary, we know that the corresponding a_m vanish, and only Fourier sines contribute to the above, as one would expect, since x(t) is now an odd function of t. The magnitudes of the c_m are the same for the original pulse train and the shifted pulse train, so the spectral energy distribution is unaffected by a time shift of the pulse train.

(b) If $\tau = T_1$, we get from (17.5) that $c_m = A\text{sinc}(m\pi)$, so now *all* lines vanish except the line at m=0, which has a coefficient A. This is again reasonable, since such a pulse train is just a constant DC function x(t) = A. In terms of Figure 17.1, the zeros spacing is now π, and *all* the delta spikes align with zeros of the sinc function, except the DC line spike.

(c) If $\tau > T_1$, the theory still applies, but the waveforms are a bit strange looking since they overlap. As τ is continuously increased, the amount of overlap builds up, and the DC coefficient continues to increase, as shown in (17.6). The black waveform below shows x(t) in the case where $T_1/\tau = 3/4$. The contributing pulses are drawn alternating red and blue and slightly displaced to make them more visible.

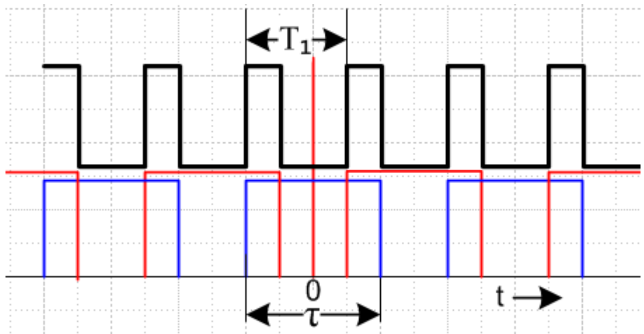


Fig 17.6

(d) If $\tau \to 0$, but $A\tau$ = area is held fixed, we have $x_{pulse}(t) \to A\tau\,\delta(t)$. For area $A\tau = 1$, $x_{pulse}(t) = \delta(t)$ and in this case x_{pulse} is exactly the first delta function model considered in Appendix A (a). To control dimensions properly, we instead set area $A\tau = T_1$ so that $x_{pulse}(t) = T_1\delta(t) = \delta(t/T_1)$ which is dimensionless. We then find from (1.1) that $X_{pulse}(\omega) = T_1$, from (17.2) that $c(\omega) = 1$, and from (17.4) the spectrum shown below,

Pulse	Pulse Train	
$x_{pulse}(t) = T_1\delta(t)$	$x(t) = \sum_{n=-\infty}^{\infty} T_1\delta(t - nT_1)$	
$X_{pulse}(\omega) = T_1$	$X(\omega) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - m\omega_1)$.	(17.11)

Thus, in the spectrum of a sequence of time-domain delta functions, all "lines" are present and have the same coefficient 2π. This function is often used as a sampling function in A/D conversion analysis. Notice that here, even though the time-domain pulse is a delta function, it's spectrum is still continuous -- being a constant T_1. The infinite pulse train spectrum is discrete, as always. We shall have more to say later on the implications of (17.11).

(e) We started with a time-domain pulse centered at t = 0. As noted earlier, if this is not the case, $X_{pulse}(\omega)$ picks up the phase exp(-iωa) where a is the new time origin of the pulse. Looking at (15.5), we see that as $x(t) \to x(t-a)$, $c_m \to e^{-im\omega_1 a}\, c_m$ so the phasor c_m simply rotates in the complex plane. This does not affect any of our qualitative conclusions above, such as lines disappearing in certain cases. Also, the DC coefficients are unaffected since this exp(-iωa) = 1 at ω = 0. As one slides the pulse train by varying point a, the mixture of real and imaginary part of the c_m varies. This corresponds to amplitude moving between the sine and cosine terms of the Fourier series. The energy/power in spectral lines is unaffected since $|c_m|^2$ does not change.

18. Non-positive pulse trains

This is pretty much a non-issue. We can take any pulse train described by coefficients c_m and superpose a constant DC level of say - B units. This corresponds to Δc_0 = -B. Thus, if we choose B = -A/2, we can cancel out the DC level in our pulse trains of (17.9) or (17.10).

Here are the c_m for a pulse train with no DC offset, 50% duty cycle, peak-to-peak amplitude A, and with falling edge aligned on t=0, taken from (17.10) with cancellation of the DC term :

$$c_m = i\,(A/\pi)\,(1/m) \qquad m = \text{odd} \qquad (18.1)$$
$$c_m = 0 \qquad m = \text{even}$$

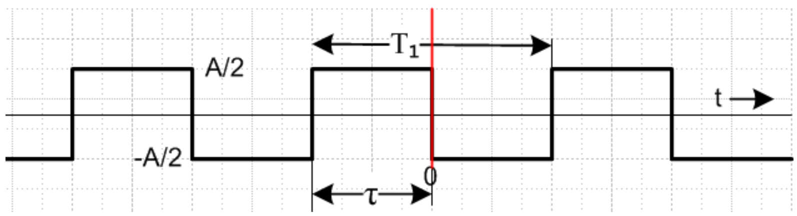


Fig 18.1

19. Biphase pulse and pulse train

Define a biphase pulse as being centered at t=0. The left pulse has width τ and amplitude A/2, the right pulse has width τ and amplitude -A/2, so the peak-to-peak amplitude is A, and a negative going edge aligns with t=0.

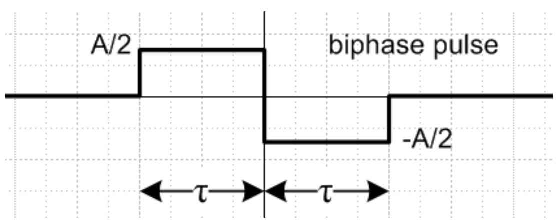


Fig 19.1

We can analyze this as the superposition of a positive and negative pulse of the square type studied above, but each pulse has amplitude A/2 instead of A. Also, the positive square pulse is time- shifted to the left by τ/2 and the negative pulse is shifted to the right by τ/2, so we pick up as corresponding spectral (12.1) "shift phase" on each contributing pulse. The result is:

$$x_{pulse}(t) = \text{SquarePulse}(A/2, t+\tau/2) - \text{SquarePulse}(A/2, t - \tau/2) \qquad (19.1)$$

$$X_{pulse}(\omega) = (A\tau/2)\,\text{sinc}(\omega\tau/2)\,[\,e^{+i\omega\tau/2} - e^{-i\omega\tau/2}]$$

$$= (A\tau)\,\text{sinc}(\omega\tau/2)\,[i\sin(\omega\tau/2)\,] = (A\tau)\,\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\,[i\sin(\omega\tau/2)\,] = (2iA/\omega)\,\sin^2(\omega\tau/2)$$

$$= (iA\tau)\,\sin^2(x)/x \qquad x = \omega\tau/2\ . \qquad (19.2)$$

This is the same as our envelope (9.2) for the positive pulse train (with pulse centered at t=0), *except* for the extra factor [isin(ωτ/2)]. Because of this extra factor, the coefficient envelope here is quite different from that for the square pulse. As $\omega \rightarrow 0$, the envelope function approaches zero -- there is no longer a central hump. Here is a normalized plot comparing the box spectrum (9.2) [red] to the biphase pulse spectrum (19.2) [black], both in absolute value :

```
plot([abs(sin(x)/x), 1/x, sin(x)^2/x], x = 0..15, y = 0..1,thickness=2,
color=[red,blue,black], tickmarks = [10,10],numpoints = 500);
```

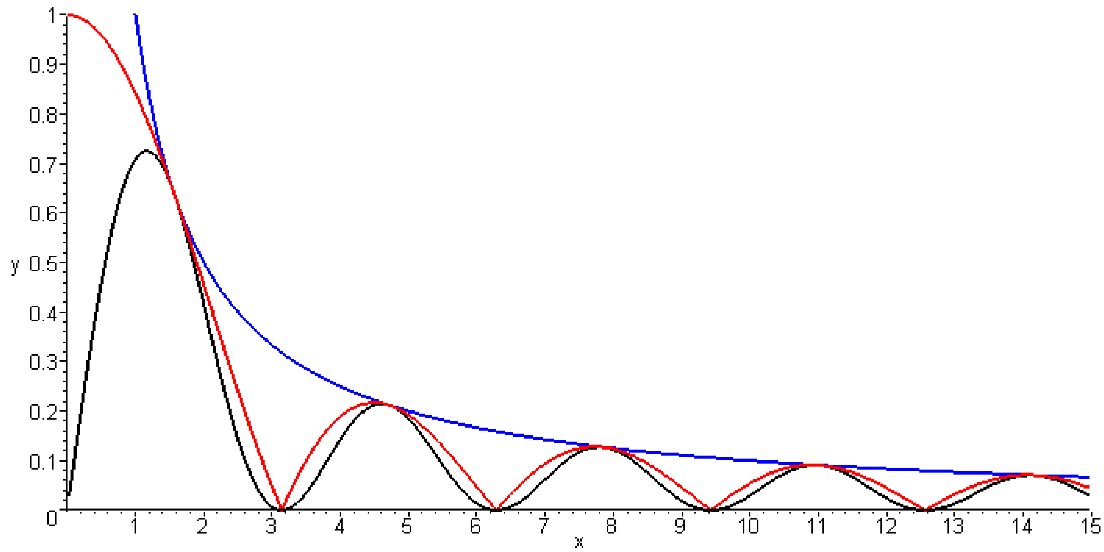


Figure 19.2. Same |sinc(x)| and 1/x function, with biphase $\sin^2(x)/x$ plot added. Fig 19.2

Our biphase pulse train spectrum is given by (17.4) and a new version of (17.5),

$$X(\omega) = \sum_{m=-\infty}^{\infty} c_m \, 2\pi\delta(\omega - m\omega_1) \qquad (17.4)$$

$$c_m = (1/T_1)\, X_{pulse}(m\omega_1) = (iA\tau/T_1)\, \text{sinc}(m\pi\tau/T_1)\, \sin(m\pi\tau/T_1)$$

$$= (iA/m\pi)\ \sin^2(m\pi\tau/T_1)\ . \qquad (19.3)$$

Notice that $c_0 = 0$ so that all biphase pulse trains have zero DC offset.

Now select the special case $\tau = T_1/2$ to construct a square wave with zero DC component,

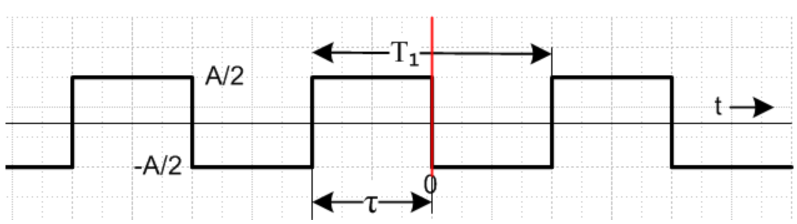


Fig 19.3

The expression (19.3) reduces to:

$$c_m = (iA/m\pi)\sin^2(m\pi/2)\ . \qquad (19.4)$$

As before, the even lines all vanish. For the odd m lines, the phase factor in (17.8) is now squared, so it is always 1. Thus, we summarize our results for a ($\tau = T_1/2$) biphase pulse train:

$$c_m = (iA/\pi m) \qquad m = \text{odd}$$
$$c_m = 0 \qquad m = \text{even} \qquad (19.5)$$

This result agrees exactly with (18.1) which was obtained by a different process involving three steps: (1) treat a positive symmetric square wave with positive pulse centered at t=0; (2) shift it so that negative going edge aligns with t=0, thus changing the phase factor; (3) add a DC term to cancel the DC offset.

One might wonder how such different spectra envelopes (black and red in Fig 19.1) can yield exactly the same c_m coefficients for m = 1,2,3... in the case $\tau = T_1/2$. The reason is easily understood. When $\tau = T_1/2$, the delta spikes in Figure 19.1 are positioned at $x = m\pi/2$, and at these points the two curves have the same values.

Here is a (semi) logarithmic view Fig 19.2. Of course log(0) = -∞, so the downward spikes of both the red and black curves really go down infinitely far, but get truncated in the plotting calculation mesh. Spectrum analyzers often allow for such a logarithmic vertical scale.

```
logplot([abs(sin(x)/x), 1/x, sin(x)^2/x], x =
1..15,color=[red,blue,black],numpoints = 500, thickness=2,xtickmarks = 10);
```

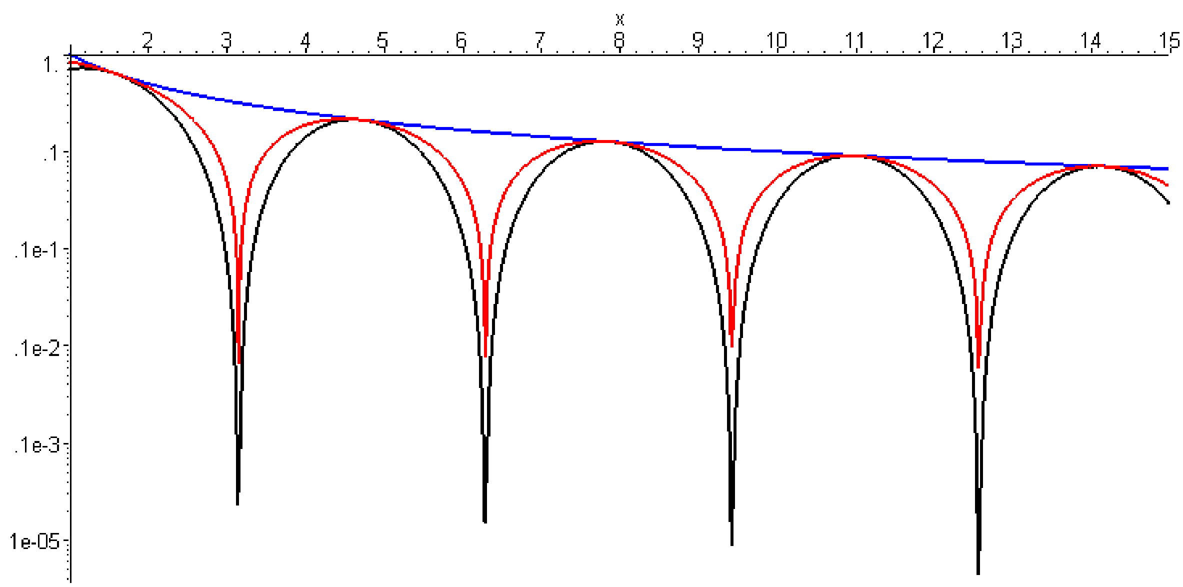


Figure 19.4. Logarithmic version of Figure 19.2. Fig 19.4

Chapter 3: Sampled Signals and Digital Transforms

In Sections 20-26, we shall used symbols Δt, T_1, t_n, and ω_1 frequently. We use whichever symbol seems most convenient at the moment. Δt is the time spacing between samples of an analog signal. We define $T_1 = \Delta t$ to make a connection with Chapter 2 where T_1 was the spacing between pulses superposed to make a pulse train. As before, $\omega_1 = 2\pi/T_1$. Symbol $t_n = n\,\Delta t$ represents the particular times we choose to examine some signal. So:

$$T_1 = \Delta t \qquad \omega_1 = 2\pi/T_1 = 2\pi/\Delta t \qquad t_n = n\,\Delta t = n\,T_1$$

20. Sampled Signals and their Image Spectra

As a specific application of our simple pulse train results boxed in (14.12) we consider the case where pulse $x_{pulse}(t)$ is a delta function, so we have a pulse train x(t) which is an infinite sequence of these delta functions spaced by time T_1. This time we let T_1 be the amplitude of each delta function, and as usual, we use $\omega_1 = 2\pi/T_1$. The basic equations for this situation are :

$$x_{pulse}(t) = T_1\,\delta(t) = \delta(t/T_1) \qquad \text{// dimensionless}$$

$$X_{pulse}(\omega) = T_1 \qquad \textit{(8.3)}$$

$$c(\omega) = 1 \qquad \textit{(14.12) item 2}$$

$$d(t) = \sum_{n=-\infty}^{\infty} T_1\,\delta(t - nT_1) \qquad \textit{(14.12) item 1} \qquad (20.1)$$

$$D(\omega) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - m\omega_1) \qquad \textit{(14.12) item 3} \qquad (20.2)$$

where we have renamed our pulse train of delta functions and its spectrum to be d(t) and D(ω), in order to free up x(t) and X(ω) for new meanings.

If we now multiply the delta function sequence d(t) times some reasonable continuous signal y(t), the result is a set of delta spikes which are **amplitude modulated** by the values that y(t) takes at the spike sampling points $t_n = nT_1$. This product we shall call x(t), it is our "sampled signal", and ω_1 is the radian/sec "sampling rate".

$$x(t) = y(t)\,d(t) = \sum_{n=-\infty}^{\infty} y(t)T_1\delta(t - nT_1) = \sum_{n=-\infty}^{\infty} y(t_n)\,T_1\delta(t - nT_1) = \sum_{n=-\infty}^{\infty} y_n\,T_1\delta(t - nT_1) \qquad (20.3)$$

where $y_n \equiv y(t_n)$ and $t_n = n\,T_1$. We can apply the "reverse" convolution theorem stated in (3.7) to the leftmost equation in (20.3) to get

$$X(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega' \, Y(\omega - \omega') \, D(\omega') \ . \tag{20.4}$$

Inserting (20.2) into (20.4) quickly yields a famous result

$$X(\omega) = (1/2\pi) \int_{-\infty}^{\infty} d\omega' \, Y(\omega - \omega') \, D(\omega') = (1/2\pi) \int_{-\infty}^{\infty} d\omega' \, Y(\omega - \omega')[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega' - m\omega_1)]$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega' \, Y(\omega - \omega') \, \delta(\omega' - m\omega_1) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1)$$

or

$$X(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) \ . \tag{20.5}$$

We have therefore shown that,

$$x(t) = \sum_{n=-\infty}^{\infty} T_1 \, \delta(t - t_n) \, y(t) = \sum_{n=-\infty}^{\infty} T_1 \, \delta(t - t_n) \, y(t_n) \tag{20.6}$$

$$X(\omega) = Y(\omega) + \sum_{m \neq 0} Y(\omega - m\omega_1) \ . \tag{20.7}$$

The first term on the right side of (20.7) is the good old Fourier Integral spectrum of y(t). The second term is a set of identical copies of Y(ω) that are shifted by all possible integer multiples of ω_1. Usually these are called **image spectra**, and the term Y(ω) is called the **main spectrum**. Here is a picture,

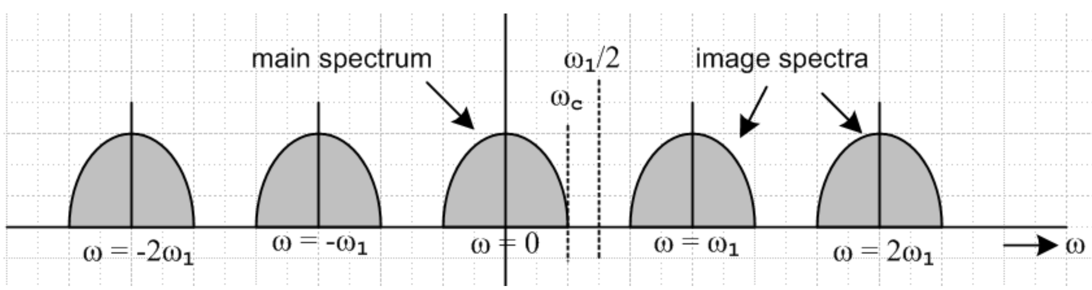


Figure 20.1. Example of a main spectrum with four of the image spectra. Fig 20.1

Thus, by sampling signal y(t) with delta functions to create the sampled signal x(t), we have picked up an infinite set of image spectra in addition to the main spectrum Y(ω).

If the spectrum Y(ω) of the "reasonable" original signal y(t) completely cuts off at some ω_c below $\omega_1/2$ (as shown in Figure 20.1), then the spectra are completely disjoint. One can then run signal x(t) through a low-pass filter that removes all these image spectra, ending up with X(ω) = Y(ω). And from Y(ω), one can presumably reconstruct y(t). Thus, these image spectra can be "dealt with". The larger the gaps between the spectra, the lower the cost of the low-pass filter required to remove the image spectra.

Of course if the spectrum of y(t) has $\omega_c > \omega_1/2$, then the spectra in (20.7) and Fig 20.1 overlap, and it is impossible to recover $Y(\omega)$ by itself using a low pass filter. If a low pass filter is placed just above the end of the $Y(\omega)$ spectrum at ω_c, the filtered signal will be contaminated with contributions from the first image spectrum, an effect loosely known as **aliasing**, as indicated in this picture,

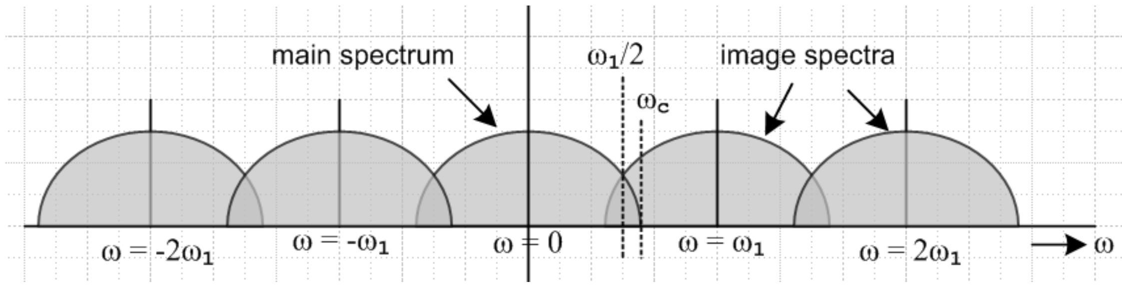


Figure 20.2. Here the image spectra overlap the main one. This is bad news. Fig 20.2

There is no place one can set a low-pass or band-pass filter to cleanly capture *just* the main spectrum (or any of its images) by itself. To avoid this problem, one must select the sampling rate $\omega_1 > 2\omega_c$. The quantity $2\omega_c$ is known as the **Nyquist rate**, so the sampling rate must be larger than the Nyquist rate. This means that for the highest frequency of interest, one must have at least 2 samples per sine wave. In audio, one thinks of $\omega_c/2\pi \approx 20$ KHz, so the Nyquist rate is $2\omega_c/2\pi$ = 40KHz and typical values for ω_1 are $\omega_1/2\pi$ = 44.1 KHz or 48 KHz. Aliasing in audio sounds like distortion, and in video causes "edge jaggies" and other artifacts.

21. Digital Filters, Image Spectra and Group Delay

(a) A Digital Filter as an approximation to an Analog Filter

In Section 3 we derived the convolution theorem stated in (3.6) which we repeat here:

$$a(t) = \int_{-\infty}^{\infty} dt'\ b(t-t')\ c(t') \quad \text{sometimes written} \quad a = b * c \tag{21.1}$$

$$A(\omega) = B(\omega)\ C(\omega) \quad . \tag{21.2}$$

As demonstrated in Section 4 (b), one can interpret c(t) as an input signal, a(t) as an output signal, and b(t) as a "filter" which acts on the input to create the output. Equation (21.2) shows the action of such a filter in the frequency domain. B(ω) might be a low-pass filter, a band-pass filter, or some other filter.

When a spectrum like B(ω) is associated with a filter, it is called the **transfer function** of that filter.

As discussed in the second comment after (3.7), the filter (21.1) can be thought of as $\mathbf{a} = \mathcal{B}\mathbf{c}$ where $\mathcal{B}$ is a linear integral operator, so the filter (21.1) is linear in the usual sense of a linear operator,

$$\mathcal{B}\,(c_1+c_2) = \mathcal{B}c_1 + \mathcal{B}c_2 \qquad \text{and} \qquad \mathcal{B}\,(\alpha c) = \alpha\mathcal{B}\,(c).$$

Moreover, by considering the fact that

$$a(t+\Delta) = \int_{-\infty}^{\infty} dt'\ b(t+\Delta-t')\ c(t') = \int_{-\infty}^{\infty} dt''\ b(t-t'')\ c(t''+\Delta) \qquad // \ -t'' = \Delta - t' \tag{21.3}$$

one sees that the filter is invariant under a time shift of the input stream. This might not be the case if the filter kernel had the more general form b(t,t') instead of b(t-t').

Filters of the type (21.1) are therefore referred to as **linear time-invariant** (**LIT**) filters.

A time-domain *digital* filter can only approximate the continuous integration shown in (21.1). What a digital (FIR) filter really does is this,

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t\ b(t_n - t_m)\ c(t_m) \qquad \text{where } t_n = n\,\Delta t\,. \tag{21.4}$$

The objects appearing in (21.4) are just numbers -- the values the functions a, b and c take at particular times, so one could just as well write this as

$$a_n = \sum_{m=-\infty}^{\infty} \Delta t\ b_{n-m}\, c_m \quad . \tag{21.5}$$

Think of the "digital filter" as a set of numbers b_k ; c_k is the input signal to the filter, and a_n is the output of the filter. Typically these numbers are represented by one byte in (black & white) video, and by two bytes in audio. The set of numbers b_k is in practice finite. For example, a "5 tap filter" has only these non-zero values: b_{-2}, b_{-1}, b_0, b_1, b_2 . Therefore the summation in (21.5) in practice is finite. Δt is the time between samples, so perhaps 1/Δt is 13.5 MHz for digital 601 video or 44.1 KHz for digital audio.

Since in digital practice the convolution integral (21.1) is replaced by the summation (21.5), one is forced to ask oneself: what happens in this case to (21.2)? We have all the tools needed to answer this question.

Recall the Fourier Integral transform pair (1.1) and (1.2) which we repeat here,

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{// projection, transform} \qquad (21.6)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \ . \qquad \text{// expansion, inverse transform} \qquad (21.7)$$

If we set $t = t_n = n\,\Delta t$, we can rewrite (21.7) as:

$$x(t_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega n\Delta t} \ . \qquad (21.8)$$

Here now is the set of steps one needs to carry out:

(1) write (21.8) for each of the functions $a(t_n)$, $b(t_n)$ and $c(t_n)$ in terms of $A(\omega)$, $B(\omega)$, and $C(\omega')$:

$$a(t_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{+i\omega n\Delta t}$$

$$b(t_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t}$$

$$c(t_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, e^{+i\omega' m\Delta t} \qquad (21.9)$$

(2) jam these three expansions into (21.4) :

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t\ b(t_n - t_m)\, c(t_m) \qquad \textit{(21.4)}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{+i\omega n\Delta t} = \sum_{m=-\infty}^{\infty} \Delta t \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega(n-m)\Delta t} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, e^{+i\omega' m\Delta t}$$

(3) move the m-summation as far to the right as possible, it comes to rest against an exponential,

$$\text{RHS} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega') \sum_{m=-\infty}^{\infty} \Delta t\, e^{+i(\omega'-\omega)m\Delta t}$$

(4) do this summation using the exponential addition theorem (13.2) with $k = \Delta t(\omega - \omega')$:

$$\sum_{m=-\infty}^{\infty} e^{im\Delta t(\omega-\omega')} = \sum_{m=-\infty}^{\infty} 2\pi\,\delta[\,\Delta t(\omega - \omega') - 2\pi m\,] = (2\pi/\Delta t) \sum_{m=-\infty}^{\infty} \delta(\,\omega - \omega' - m\omega_1)$$

Right here is where the image spectra described below first appear! Both sides of this equation treated as a function of ω are *periodic* with period ω_1. If we were to multiply both sides by Δt then take the limit $\Delta t \rightarrow 0$ we would get (since $\omega_1 = 2\pi/\Delta t$, this means $\omega_1 \rightarrow \infty$ as well)

$$\int_{-\infty}^{\infty} dt\, e^{it(\omega-\omega')} = 2\pi\,\delta(\omega-\omega')$$

which is just (2.1). The images (δ lines at this point) have run off to infinity and the function is no longer periodic. So image spectra arise from the fact that a discrete sum of phasor functions $e^{im\Delta t(\omega-\omega')}$ (each of which is periodic in ω) produces a periodic function, even if that sum is infinite.

(5) kill the $d\omega'$ integration against the delta function

$$\begin{aligned} \text{RHS} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, 2\pi \sum_{m=-\infty}^{\infty} \delta(\,\omega - \omega' - m\,\omega_1) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, \delta(\,\omega - \omega' - m\,\omega_1) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t} \sum_{m=-\infty}^{\infty} C(\omega - m\omega_1) \end{aligned}$$

so that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{+i\omega n\Delta t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t} \sum_{m=-\infty}^{\infty} C(\omega - m\omega_1) \; .$$

Since the functions $e^{+i\omega n\Delta t}$ form a complete set, we may identify the integrands to obtain

$$A(\omega) = B(\omega) \sum_{m=-\infty}^{\infty} C(\omega - m\omega_1)\; . \tag{21.10}$$

(6) Alternatively, we could kill the $d\omega$ integration against the delta function. We repeat the first line in (3) changing the order of integration

$$\begin{aligned} \text{RHS} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t}\, 2\pi \sum_{m=-\infty}^{\infty} \delta(\,\omega - \omega' - m\,\omega_1) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, C(\omega') \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{+i\omega n\Delta t}\, \delta(\,\omega - \omega' - m\,\omega_1) \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \, C(\omega') \sum_{m=-\infty}^{\infty} B(\omega' + m\omega_1) \, e^{+i(\omega' + m\omega_1 n)\Delta t} \ .$$

But $e^{im\omega_1 n\Delta t} = 1$ because $m\omega_1 n\Delta t = mn(2\pi/T_1)T_1 = mn2\pi$. Since the m sum is symmetric, we can replace m→ -m making no difference, and we then replace ω'→ω on the RHS. The result is then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, A(\omega) \, e^{+i\omega' n\Delta t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, C(\omega) e^{i\omega\Delta t} \sum_{m=-\infty}^{\infty} B(\omega - m\omega_1) \ .$$

Again using the completeness of the $e^{+i\omega n\Delta t}$ basis functions, we equate integrands to get

$$A(\omega) = \sum_{m=-\infty}^{\infty} B(\omega - m\omega_1) \ C(\omega) \ . \qquad (21.11)$$

which is the form we shall use below. We noted in (3.2) how the convolution theorem is invariant under b↔c, and we see this symmetry in the two results just obtained, (21.10) and (21.11). We have then arrived at this statement of our *digital* convolution theorem:

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t \ b(t_n - t_m) \, c(t_m) \qquad t_n = n \, \Delta t \qquad (21.12)$$

$$A(\omega) = [\, B(\omega) + \sum_{m \neq 0} B(\omega - m\omega_1) \,] \, C(\omega) \ . \qquad (21.13)$$

This pair of equations should be compared to the analog convolution theorem (21.1) and (21.2),

$$a(t) = \int_{-\infty}^{\infty} dt' \, b(t-t') \, c(t') \quad \text{sometimes written} \quad a = b * c \qquad \mathit{(21.1)}$$

$$A(\omega) = B(\omega) \, C(\omega) \qquad \mathit{(21.2)}$$

We see that there is a *penalty* for working in the imperfect, discrete world of time-sampled signals like $a(t_n)$. The penalty is that there are extra ω-space terms in (21.13) that are not present in (21.2).

Recall that B(ω) is the spectrum (transfer function) of a filter kernel b(t) which we are approximating by a set of coefficients b_k. In analogy with the spectrum X(ω) shown in (20.7) of a delta-sampled signal x(t), the main term B(ω) in (21.13) is called the main spectrum of the filter, while the other terms in the square bracket are the filter's image spectra passbands. The filter then has a transfer function which looks like the spectrum of Fig 20.1.

A perfect analog filter would of course have no such image spectra, this is what (21.2) is all about. It has no such artifacts because it filters at all times t, not just at particular points t_n. The digital filter is "blind" between sample points, so you can stick it with some high frequency signals which wiggle an arbitrary number of wiggles between the sample points. These high frequency signals are what in effect get passed through the image pass bands of a digital low-pass filter. All the above math should not blind the reader to this straightforward physical understanding of the image spectra. Here is an example,

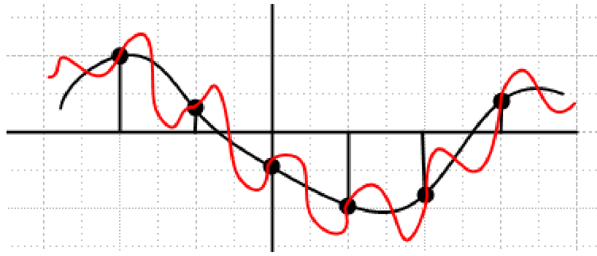


Fig 21.1

The red and black signals are treated exactly the same by a digital filter since they have exactly the same sample values. But the red signal has a very strong frequency component with period $\Delta t = T_1$ and thus with frequency $\omega_1 = 2\pi/T_1$ and so the red signal in effect passes through the first image passband of the filter, giving the same output signal that the black signal would give going through the main passband. One says that the red signal is an **alias** of the black signal, or it is **aliased** into the black signal, giving the same filter output. Perhaps a violin comes of our filter out sounding like a tuba.

A common use of a digital filter is to remove the image spectra of digitized signals. These image spectra are sitting staring us in the face in (20.7). Suppose we construct a digital filter with some set of b_n coefficients to implement a low-pass filter to remove the signal image spectra. But we have just seen that this filter itself has image passbands, so we have to be careful that some of the image spectra of our sampled signal x(t) don't slip through these image pass bands of the filter. A standard trick ("oversampling") is to run the filter at a rate ω'_1 which is perhaps 4X or 8X times faster than the rate ω_1 of the sampled signal ($\omega_1 = 2\pi/\Delta t$). Recall that $\omega_1 > 2\omega_c$, the Nyquist rate. The filter's image spectra now at $m\omega_1'$ are then pushed away from the central region, causing the lower image spectra of the signal x(t) to be blocked by the filter. Some very high frequency data might get through the image bands of the filter, but this can be removed by a simple analog filter (perhaps just a resistor and capacitor) after the D/A converter which converts the digital signal to analog. An example of this technique is presented in Section 30 using a digital filter implemented in Section 29 which has the desirable properties of Section 28.

(b) Filter Group Delay

The discussion here is given in terms of an analog filter. The steps stated below can be repeated for a digital filter and one arrives at the same set of conclusions.

Recall the convolution theorem from the start of this section,

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')\, c(t') \qquad \text{sometimes written} \qquad a = b * c \tag{21.1}$$

$$A(\omega) = B(\omega)\, C(\omega) \quad . \tag{21.2}$$

We interpret this as a filter acting on signal c(t) to produce signal a(t). To assist this interpretation, we rename signals in this way (i = input, o = output)

$$o(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')\, i(t') \quad \text{sometimes written} \quad o = b * i \qquad (21.1)$$

$$O(\omega) = B(\omega)\, I(\omega) \qquad (21.2)$$

where b and B represent the action of the filter. In general we can write the complex filter spectrum in terms of its magnitude and phase functions (using a traditional sign convention for said phase)

$$B(\omega) = |B(\omega)|\, e^{-i\varphi(\omega)} \ . \qquad (21.14)$$

In order to derive the concept of group delay, we assume that our filter is a passband filter of width 2a centered at some frequency ω_2, and having this somewhat idealized spectral shape,

$$B(\omega) = \begin{cases} |B(\omega)|\, e^{-i\varphi(\omega)} & \text{if } \omega_2\text{-a} < \omega < \omega_2\text{+a} \\ 0 & \text{outside this narrow band} \end{cases} \qquad (21.15)$$

This could for example be a low-pass filter centered at $\omega_2 = 0$ with ω range (-a,a).

Let us assume that i(t) represents a very narrow input pulse whose center lies at t = 0. Since the pulse is narrow in the time domain, we know (uncertainty principle in Section 1) that it will have a broad smoothly-varying spectrum $X_{pulse}(\omega)$. The ultimate pulse is $i(t) = \delta(t)$ which has $X_{pulse}(\omega) = 1$ from (8.3).

From (1.2) the output of the filter can be written as

$$o(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, O(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, B(\omega)\, I(\omega)\, e^{+i\omega t}$$

$$= (1/2\pi) \int_{-\infty}^{\infty} d\omega\, B(\omega)\, X_{pulse}(\omega) e^{+i\omega t} = (1/2\pi) \int_{\omega_2-a}^{\omega_2+a} d\omega\, |B(\omega)|\, X_{pulse}(\omega) e^{-i\varphi(\omega)} e^{+i\omega t}$$

$$\approx (1/2\pi)\, |B(\omega_2)|\, X_{pulse}(\omega_2) \int_{\omega_2-a}^{\omega_2+a} d\omega\, e^{-i\varphi(\omega)} e^{+i\omega t} \ .$$

We have assumed that $|B(\omega)|$ is a smooth function near $\omega = \omega_2$ to make the approximation on the last line. Similarly, we assume that that filter phase function is also smooth so we can approximate it in this linear fashion in the neighborhood of $\omega = \omega_2$,

$$\varphi(\omega) \approx \varphi(\omega_2) + (\omega-\omega_2)\varphi'(\omega_2) = \alpha + (\omega-\omega_2)\beta \qquad \alpha = \varphi(\omega_2) \qquad \beta = \varphi'(\omega_2) \ . \qquad (21.16)$$

Then we find that

$$o(t) \approx (1/2\pi)\, |B(\omega_2)|\, X_{pulse}(\omega_2)\, e^{-i(\alpha-\beta\omega_2)} \int_{\omega_2-a}^{\omega_2+a} d\omega\, e^{+i\omega(t-\beta)} \ .$$

The integral may be evaluated as

$$\int_{\omega_2-a}^{\omega_2+a} d\omega\, e^{+i\omega(t-\beta)} = [i(t-\beta)]^{-1} [\, e^{+i(\omega_2+a)(t-\beta)} - e^{+i(\omega_2-a)(t-\beta)}]$$

$$= [i(t-\beta)]^{-1} e^{i\omega_2(t-\beta)}\ 2i \sin[a(t-\beta)] \quad = 2a\, e^{i\omega_2(t-\beta)} \sin[a(t-\beta)]/ [a(t-\beta)]$$

$$= 2a\, e^{i\omega_2(t-\beta)} \operatorname{sinc}[a(t-\beta)] \,. \qquad (21.17)$$

Thus, the filter output is

$$o(t) = (a/\pi)\, |B(\omega_2)|\, X_{pulse}(\omega_2)\ e^{-i(\alpha-\beta\omega_2)}\ e^{i\omega_2(t-\beta)} \operatorname{sinc}[a(t-\beta)]$$

$$= [(a/\pi)\, |B(\omega_2)|\, X_{pulse}(\omega_2)\ e^{-i\alpha}]\ e^{i\omega_2 t} \operatorname{sinc}[a(t-\beta)] \,. \qquad (21.18)$$

The last two factors show the time dependence of o(t). The $e^{i\omega_2 t}$ represents an oscillation at ω_2 which is the center of the bandpass filter. This is modulated by an envelope function sinc[a(t-β)] causing the spectrum of o(t) to fill the pass band, as we also know from O(ω) = B(ω) I(ω) . The initial narrow pulse i(t) = X_{pulse}(t) is spread out into a pulse o(t) of width determined by the first zero of the sinc function, and centered at t = β. Comparing the center of the input and output pulses, one concludes that the pulse has been delayed by amount β, which is known as the group delay. Recall that β = φ'(ω_2) .

We have therefore proven the following theorem:

Group Delay Theorem. When a narrow time-domain pulse is passed through a bandpass filter, the output pulse is delayed approximately by an amount τ_d = dφ/dω evaluated at the bandpass center frequency. This delay is called the group delay of the filter.

(21.19)

Corollary. If the phase function of a filter φ(ω) is linear in ω, then the phase approximation made in (21.16) is exact, so the theorem just stated has a group delay which is a constant throughout the passband of the filter. That is to say, dφ/dω is a constant for a filter with linear phase. The implication is that different pulse shapes, each having slightly different spectra, will all pass through the filter with the same delay, regardless of where in the bandpass band these pulse spectra hit. The result is good "fidelity" of a time varying signal such as an audio signal or a radar pulse stream.

(21.20)

22. The Digital Fourier Transform X'(ω) Part I

As a reminder from the opening paragraph of this Chapter,

$$T_1 = \Delta t \qquad \omega_1 = 2\pi/T_1 = 2\pi/\Delta t \qquad t_n = n\,\Delta t = n\,T_1 \ .$$

Recalling from (1.1) that $X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t}$, we can write down the following *non*-equation:

$$X(\omega) \neq \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t} \qquad \text{projection = transform} \qquad (22.1)$$

X(ω) is the genuine Fourier Integral transform of x(t). Only in the limit $\Delta t \to 0$ are the two sides equal, and we then reproduce (1.1). So let's define something new called X' that *is* equal for any finite Δt:

$$X'(\omega) \equiv \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t} \qquad \text{projection = transform} \qquad (22.2)$$

Dimensions: If $\text{Dim}[x(t_n)] = V$, then $\text{Dim}[X'(\omega)] = V\text{-sec}$, the same as $\text{Dim}[X(\omega)]$.

Although X(ω) can have any shape we want, the new spectrum X'(ω) is periodic with period ω_1 ,

$$X'(\omega - m\omega_1) = \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i(\omega - mn\omega_1)\Delta t} = \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t} = X'(\omega)$$

where, as earlier, $e^{im\omega_1 n\Delta t} = 1$ because $m\omega_1 n\Delta t = mn(2\pi/T_1)\, T_1 = mn2\pi$. So X'(ω) is periodic:

$$X'(\omega - m\omega_1) = X'(\omega) \qquad m = \text{any integer} \ . \qquad (22.3)$$

We now claim (to be shown below) that the inverse of (22.2) is the following:

$$x(t_m) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega m\Delta t} \qquad \text{expansion = inversion} \qquad (22.4)$$

This looks like to (1.2) except the integration endpoints are here finite. Thus, we have a *different* projection formula, and a correspondingly *different* expansion formula. For want of a better name, let us call this new transform the **Digital Fourier Transform** pair, as opposed to the Fourier Integral Transform pair given in (21.6) and (21.7).

We shall now verify that (22.4) is correct "in both directions". We do this in full detail to give the reader a chance to "practice" using many results presented earlier.

First, insert (22.4) into the right side of (22.2) to get ,

$$\sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t} = \sum_{n=-\infty}^{\infty} \Delta t\, [\frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X'(\omega')\, e^{+i\omega' n\Delta t}]\, e^{-i\omega n\Delta t}$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X'(\omega)\, \Delta t \sum_{n=-\infty}^{\infty} e^{+in\Delta t(\omega'-\omega)} \qquad \text{// sliding n sum to the right}$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X'(\omega)\, \Delta t \sum_{m=-\infty}^{\infty} 2\pi\delta(\Delta t(\omega'-\omega) - 2\pi m) \qquad \text{// (13.2) with } k = \Delta t(\omega'-\omega)$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X'(\omega')\, \delta(\omega'-\omega-m\omega_1) \qquad \text{// } \delta(ax) = (1/a)\delta(x)$$

$$= \sum_{m=-\infty}^{\infty} X'(\omega+m\omega_1)\ \Theta(-\omega_1/2 \le \omega+m\omega_1 \le \omega_1/2) \qquad \text{// (2.2) with special } \Theta \text{ notation}$$

$$= X'(\omega) \sum_{m=-\infty}^{\infty} \Theta(-\omega_1/2 \le \omega+m\omega_1 \le \omega_1/2) \qquad \text{// (22.3) that } X'(\omega + m\omega_1) = X'(\omega)$$

$$= X'(\omega) \quad . \qquad \text{// (A.50), see Appendix A (e).} \qquad (22.5)$$

In the second last step, we used (22.3) that $X'(\omega+m\omega_1) = X'(\omega)$, allowing $X'(\omega)$ to be extracted from the sum on m. Then in the last step we use (A.50) with $\alpha = \omega_1$ and $x = \omega$. The sum $\Sigma_m \Theta = 1$ basically says that one partitions the curve $f(\omega) = 1$ into little sections of length ω_1, with attention paid to what happens at the boundaries of these little sections.

Second, insert (22.2) into the right side of (22.4) to get

$$\frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega m\Delta t} = \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, [\sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t}]\, e^{+i\omega m\Delta t}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta t\, x(t_n) \int_{-\omega_1/2}^{\omega_1/2} d\omega\ e^{-i\omega(n-m)\Delta t} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \Delta t\, x(t_n) \int_0^{\omega_1/2} d\omega\, \cos[(n-m)\Delta t\omega]$$

$$= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, \delta_{n,m}\ (\pi/\Delta t) = \sum_{n=-\infty}^{\infty} x(t_n)\, \delta_{n,m} = x(t_m)$$

Here we have used $\int_0^{\omega_1/2} d\omega\, \cos[(n-m)\Delta t\omega] = \delta_{n,m}\, (\pi/\Delta t)$ since

$$\int_0^{\omega_1/2} d\omega \cos[(n-m)\Delta t\omega] = \frac{1}{(n-m)\Delta t} \sin[(n-m)\,\Delta t\,(\omega_1/2)\,] = 0 \qquad n \neq m$$

$$\int_0^{\omega_1/2} d\omega \cos[(n-m)\Delta t\omega] = \int_0^{\omega_1/2} d\omega = (\omega_1/2) = (\pi/\Delta t) \qquad n = m$$

One should keep in mind that this new transform, the Digital Fourier Transform, is dependent on the constant $\Delta t = T_1$. Changing this constant changes the transform. The Fourier Integral Transform contains no such constant. In effect, $T_1 = 0$.

We use the term "digital" in Digital Fourier Transform only because in the time domain the function x(t) is represented by a sequence of evenly spaced samples $x(t_n)$ and we say nothing about what x(t) might be doing between these sample times. The term Digital Fourier Transform is just our unofficial name for this transform, and the official name will appear later in Section 24.

Notice that both the Fourier Integral Transform and the Digital Fourier Transform are (at first) used to analyze time-domain functions (or sequences) that are of limited temporal extent, so we think of x(t) or $x(t_n)$ more or less as some kind of pulse. Technically, x(t) or $x(t_n)$ are non-periodic (aperiodic). Here is a side by side comparison of these two transforms:

Fourier Integral Transform

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

Digital Fourier Transform

$$X'(\omega) \equiv \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega n\Delta t} \qquad \text{projection = transform} \qquad (22.2)$$

$$x(t_m) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega m\Delta t} \qquad \text{expansion = inversion} \qquad (22.4)$$

For an aperiodic temporal function x(t) or sequence $x(t_n)$, the spectra X(ω) and X'(ω) are *both* continuous spectra, even though (1.1) shows X(ω) as an integral and (22.2) shows X'(ω) as a sum of functions which are continuous in ω. In the next section, we shall see the fascinating relationship between X(ω) and X'(ω).

In the limit $\Delta t \to 0$, $X'(\omega) \to X(\omega)$ and $\omega_1 \to \infty$, so the Digital Fourier Transform is where the Fourier Integral Transform ends up if the continuum of time is divided into discrete chunks.

Now let's go back to our discrete convolution relation (21.4),

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t\; b(t_n - t_m)\, c(t_m) \qquad t_n = n\,\Delta t \quad . \qquad (22.6)$$

Dimensions: $\dim(b) = \sec^{-1}$, $\dim(\Delta t\, b) = 1$, so $\dim(a) = \dim(c)$.

What does this look like in the frequency domain? To find out, we insert into (22.6) expansions of the form (22.4) for the functions a, b and c. We just did this in Section 21 above. The steps (1),(2),(3),(4) are exactly the same *except* our dω and dω' integration endpoints are $(-\omega_1/2, \omega_1/2)$ instead of $(-\infty,\infty)$. The first new feature occurs in step (5) where we pick up the analysis:

$$\frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, A'(\omega)\, e^{+i\omega n\Delta t} = \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, B'(\omega)\, e^{+i\omega n\Delta t}\, \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega'\, C'(\omega')\, 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega' - m\,\omega_1)$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, B'(\omega)\, e^{+i\omega n\Delta t} \sum_{m=-\infty}^{\infty} \int_{-\omega_1/2}^{\omega_1/2} d\omega'\, C'(\omega')\, \delta(\omega - \omega' - m\,\omega_1)$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, B'(\omega)\, e^{+i\omega n\Delta t} \sum_{m=-\infty}^{\infty} C'(\omega - m\omega_1)\, \Theta(-\omega_1/2 \le \omega - m\omega_1 \le \omega_1/2) \qquad // (2.2)$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, B'(\omega)\, e^{+i\omega n\Delta t}\, C'(\omega) \sum_{m=-\infty}^{\infty} \Theta(-\omega_1/2 \le \omega - m\omega_1 \le \omega_1/2) \quad // \text{ (22.3) for } C'(\omega)$$

$$= \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, B'(\omega)\, e^{+i\omega n\Delta t}\, C'(\omega) \qquad // \text{ (A.50), see Appendix A (e).}$$

Since $e^{+i\omega n\Delta t}$ forms a complete set on the interval $(-\omega_1/2, \omega_1/2)$, we may equate integrands to find

$$A'(\omega) = B'(\omega)C'(\omega) . \qquad (22.7)$$

Thus, our new Digital Fourier transform yields this simple diagonalized result with none of those extra image terms. Of course we must remain aware that A'(ω) is not the genuine spectrum of a(t), it is some new thing. Just because we defined a new animal and got (22.7) does not mean that the image spectra go away in (20.7) and (21.10) and (21.11). Here then is the Digital Fourier Transform convolution theorem in comparison with that for the Fourier Integral Transform:

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t\ b(t_n - t_m)\, c(t_m) \quad \Leftrightarrow \quad A'(\omega) = B'(\omega)C'(\omega) \qquad t_n = n\Delta t \qquad (22.8)$$

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \quad \Leftrightarrow \quad A(\omega) = B(\omega)\, C(\omega) \qquad \mathit{(3.6)}$$

23. The Digital Fourier Transform X'(ω) Part II

(a) Relation between X'(ω) and X(ω)

The next problem is to figure out how the genuine Fourier Integral spectrum X(ω) is related to our new Digital Fourier Transform X'(ω). Start with the Fourier Integral expansion (1.2) ,

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \quad . \qquad (1.2)$$

Partition the integration into a set of little ranges of width ω_1 :

$$x(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{m\omega_1-\omega_1/2}^{m\omega_1+\omega_1/2} d\omega\, X(\omega)\, e^{+i\omega t} \ .$$

Change integration variable to $\omega' = \omega - m\omega_1$,

$$x(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X(\omega'+m\omega_1)\, e^{+i(\omega'+mt\omega_1)}$$

$$= \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega' \left[\sum_{m=-\infty}^{\infty} X(\omega'+m\omega_1)\, e^{+im\omega_1 t} \right] e^{+i\omega' t} \ .$$

Next, set $t = t_n = nT_1$ on both sides. This makes the exponential inside the square bracket equal 1, so

$$x(t_n) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega' \left[\sum_{m=-\infty}^{\infty} X(\omega'-m\omega_1) \right] e^{+i\omega' n\Delta t} \ .$$

Now compare this to the Digital Fourier expansion defined in (22.4) which we duplicate here, changing ω to ω' and m to n:

$$x(t_n) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega'\, X'(\omega')\, e^{+i\omega' n\Delta t} \ . \qquad \text{expansion = inversion} \qquad (22.4)$$

Since the functions $e^{+i\omega' n\Delta t}$ form a complete basis on the interval $(-\omega_1/2,\omega_1/2)$, equate integrands of the last two equations to get,

$$X'(\omega) = \sum_{m=-\infty}^{\infty} X(\omega - m\omega_1) = [\ X(\omega) + \sum_{m \neq 0} X(\omega - m\omega_1)\ ] \quad . \qquad (23.1)$$

This is that thing that keeps popping up everywhere -- the main spectrum plus all the image spectra. Thus, we have shown that this combination is precisely the Digital Fourier Transform spectrum. We can therefore go back and reexamine some of our earlier results with this new knowledge:

Consider (20.7):

$$X(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) = [\ Y(\omega) + \sum_{m \neq 0} Y(\omega - m\omega_1)\ ] = Y'(\omega) \quad . \qquad (23.2)$$

This says that the Fourier Integral spectrum of a "reasonable" signal y(t) *multiplied* by a sequence of delta functions is exactly the Digital Fourier Transform spectrum Y'(ω). Of course Y'(ω) is computed from (22.2) from a knowledge of y(t) only at the sample points t_n.

Next, we realize that our digital filter equations (21.10) and (21.11) ,

$$A(\omega) = B(\omega) \sum_{m=-\infty}^{\infty} C(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} B(\omega - m\omega_1)\ C(\omega) ,$$

become

$$A(\omega) = B(\omega)\, C'(\omega)$$
$$A(\omega) = B'(\omega)\, C(\omega) \qquad (23.3)$$

The second equation is our low-pass digital filter B with input C and output A. The filter with all its image pass bands is now conveniently represented by B'(ω). A(ω) and C(ω) are still the Fourier Integral spectra of a and c. However, we already know from (22.7) that (23.3) is true with primes on A and C as well,

$$A'(\omega) = B'(\omega)\, C'(\omega). \qquad (23.4)$$

It might seem unusual that (23.3) and (23.4) can all be true. They *are* all true, and we can now present a much more compact derivation of (23.4) by making use of (23.3) :

$$A(\omega) = B'(\omega)\, C(\omega) \qquad \text{// (23.3) which is really just (21.11)}$$

$$A(\omega - m\omega_1) = B'(\omega - m\omega_1)\, C(\omega - m\omega_1) \qquad \text{// set } \omega \rightarrow \omega - m\omega_1,$$
$$= B'(\omega)\, C(\omega - m\omega_1) \qquad \text{// (22.3) for B'(}\omega\text{)}$$

Then:

$$\sum_{m=-\infty}^{\infty} A(\omega - m\omega_1) = B'(\omega) \sum_{m=-\infty}^{\infty} C(\omega - m\omega_1)$$

or

$$A'(\omega) = B'(\omega)\, C'(\omega)$$

(b) Summary of the Digital Fourier Transform

We now summarize what we know about the Digital Fourier Transform (this box takes 2 pages)

Digital Fourier Transform (23.5)

1. Let x(t) be any reasonable function.

2. Divide up the time axis into steps $t_n = n\,\Delta t$; let $T_1 = \Delta t$ and $\omega_1 = 2\pi/T_1$.

3. We can think of samples $x_n = x(t_n)$ for the above x(t). Alternatively, we can think of the x_n as some given sequence, and one could then construct an infinite number of functions x(t) for which $x_n = x(t_n)$.

4. In terms of x(t), the Digital Fourier Transform and its inverse are given by

$$X'(\omega) \equiv \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega t_n} \qquad \text{projection = transform} \qquad \textit{(22.2)} \quad \text{// V-sec}$$

$$x(t_n) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega t_n} \qquad \text{expansion = inversion} \qquad \textit{(22.4)} \quad \text{// V}$$

More generally, dispensing now with x(t) and writing $t_n = nT_1$ in the exponential,

$$X'(\omega) \equiv T_1 \sum_{n=-\infty}^{\infty} x_n\, e^{-i\omega n T_1} \qquad \text{projection = transform}$$

$$x_n = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega n T_1} \qquad \text{expansion = inversion}$$

If $\text{Dim}(x_n) = V$, then $\text{Dim}(X') = V\text{-sec}$.

5. By its definition (and $t_n = n\,\Delta t$), $X'(\omega)$ is periodic in ω with period ω_1:

$$X'(\omega - m\omega_1) = X'(\omega) \qquad m = \text{any integer} \qquad \textit{(22.3)}$$

6. The relation between $X'(\omega)$ and the Fourier Integral spectrum $X(\omega)$ of x(t) is given by:

$$X'(\omega) = \sum_{m=-\infty}^{\infty} X(\omega - m\omega_1) = [\ X(\omega) + \sum_{m \neq 0} X(\omega - m\omega_1)\ ] \qquad \textit{(23.1)}$$

7. The Digital Fourier Transform diagonalizes any convolution sum:

$$a(t_n) = \sum_{m=-\infty}^{\infty} \Delta t \; b(t_n - t_m)\, c(t_m) \qquad t_n = n\,\Delta t \qquad (22.6)$$

$$A'(\omega) = B'(\omega)\, C'(\omega) \qquad (22.7)$$

8. It is also true that

$$A(\omega) = B'(\omega)\, C(\omega) = B(\omega) C'(\omega) \qquad (23.3)$$

24. The Z Transform X"(z)

The Digital Fourier Transform described in Sections 22 and 23 is really the Z Transform times Δt. We have concealed this fact up till now because the ω-space version, called X'(ω) in Section 23, allows direct comparison to the Fourier Integral spectrum X(ω). We have already drawn the major conclusions. Here we just change the clothing.

Change variables from ω to dimensionless z, [$\omega_1 = 2\pi/\Delta t$ so $\Delta t = 2\pi/\omega_1 = \pi/(\omega_1/2) = T_1$]

$$z \equiv e^{i\omega\Delta t} = e^{i\pi[\omega/(\omega_1/2)]} \qquad dz = z\, i\, \Delta t\, d\omega \qquad d\omega = \frac{dz}{iz\,\Delta t} \ . \tag{24.1}$$

Note that as ω runs over its range $-\omega_1/2$ to $+\omega_1/2$, phasor z runs from $-\pi$ to π on a unit circle.

Now define the Z Transform X"(z) in terms of the Digital Fourier Transform X'(ω),

$$X''(z) \ \equiv \ \frac{1}{\Delta t} \ X'(\omega(z)) \ . \tag{24.2}$$

Dimensions: If $\mathrm{Dim}(x_n) = V$, then Dim(X') = V-sec so Dim(X") = V, the same as x_n, see also (24.3).

With this substitution, and letting

$$x_n = x(t_n) = x(n\Delta t),$$

the above Digital Fourier Transform formulas (22.2) and (22.4) (see box above) immediately become:

$$X''(z) = \sum_{n=-\infty}^{\infty} x_n\, z^{-n} \qquad \text{projection = transform} \tag{24.3}$$

$$x_n = \frac{1}{2\pi i} \int_C dz\, X''(z)\, z^{n-1} \qquad \text{expansion = inversion} \tag{24.4}$$

where C is a contour doing one counterclockwise traversal of the unit circle in the z plane. These two equations are **the Z Transform** and its inverse.

Limit Comment: Since X'(ω) → X(ω) as $\Delta t \to 0$, it follows that $\lim_{\Delta t\to 0} [\ \Delta t\ X''(z)] = X(\omega)$. So this is how one could get from the Z Transform to the Fourier Integral Transform.

Mapping Comment: One can think of $z = e^{i\Delta t\omega}$ as describing an analytic mapping (conformal map) from the complex ω-plane to the z-plane. Here is a picture of that mapping :

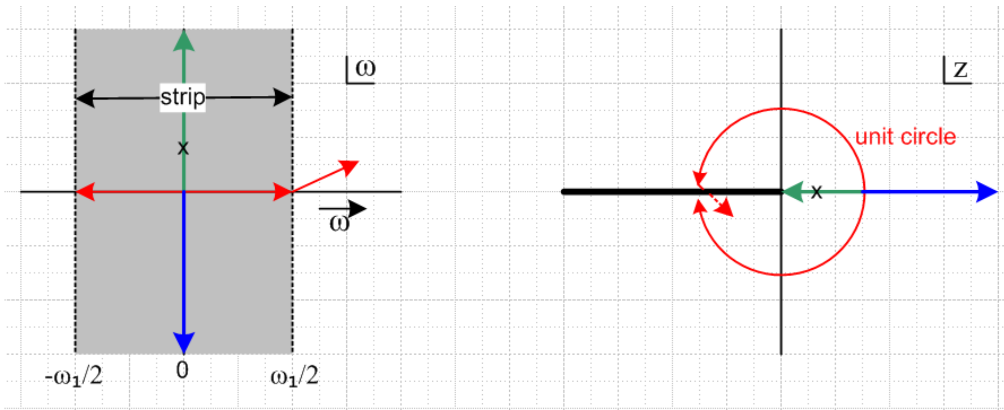


Fig 24.1

The infinite gray vertical strip of the ω-plane with $-\omega_1/2 \le \mathrm{Re}(\omega) \le \omega_1/2$ maps into the *entire* z plane. The real axis in the red range $-\omega_1/2 \le \omega \le \omega_1/2$ maps into the unit circle in the z plane as shown. The upper half of the strip in the ω-plane maps into the interior of the unit circle, and the lower half of the strip maps into the exterior of the unit circle. The blue and green arrows map as shown. Generally, horizontal segments in ω map into origin-centered circles in z, and vertical lines in ω map into rays in z.

The z plane shows the principle Riemann sheet of the mapping with a black branch cut going off to the left from the z plane origin. If in the ω plane one continues the red arrow into the next strip to the right, $\mathrm{Re}(\omega) > \omega_1/2$, one dives through the branch cut on the right and arrives on the next sheet in z.

For some general function f(ω) define $F(z) \equiv f(\omega(z))$. The function F(z) would have an induced branch cut as shown on the right, with some discontinuity across it. In this case, the red circle would not represent a closed integration contour, so the usual rules of complex integration around closed loops would not apply. However, if f(ω) were periodic with period ω_1, there would be no discontinuity across the branch cut in F(z) because f(ω) would take the same value on the two vertical edges of the grey strip in the ω plane, and therefore F(z) would have the same value on the two sides of the cut, which means there is no cut. In this case, the red circle does represent a closed contour.

According to (22.3), the Digital Fourier Transform X'(ω) is periodic with period ω_1. Therefore the Z Transform X"(z) has no branch cut and the red circle is a closed contour, as used in the examples below. This is why the Z Transform is so useful. The redundant information in the infinite number of vertical strips in the ω plane is reduced to non-redundant information in the z plane. The mapping is completely analytic, introducing no poles or branch cuts.

If a function $F(z) = (z-a)^{-1}$ has a pole in the z plane at a, as shown in Fig 24.1 by the x on the right, then f(ω) has a pole at $\omega_a = \ln(a)/(i\Delta t)$ as shown by the x on the left, $\ln(a) < 0$. To show this, consider ω in the neighborhood of ω_a:

$$f(z(\omega)) = \frac{1}{z-a} = \frac{1}{e^{i\Delta t(\omega-\omega_a)}\, e^{i\Delta t\omega_a} - a} = \frac{1}{e^{i\Delta t(\omega-\omega_a)}a - a} = \frac{1}{a(e^{i\Delta t(\omega-\omega_a)} - 1)} \approx \frac{1}{ai\Delta t(\omega-\omega_a)} .$$

The bottom line is that in going from the Digital Fourier Transform to the Z Transform, we are simply making a change of variable and removing redundant information. There is nothing dramatically new introduced by doing this. As we shall see below, there is a notational economy in writing z^{-1} for a delay of Δt in place of $e^{-i\Delta t\omega}$.

So far in these notes we have run into the Fourier Integral Transform and its Sine and Cosine cousins, the Laplace Transform, the Fourier Series Transform, the Digital Fourier Transform, and the Z Transform. They are all variations on the same theme, and we have shown how they are all related to each other. They all have an analogous set of "basic results" and "rules". We now peruse these results and rules for the Z Transform.

(a) Convolution Theorem

According to the definition $X''(z) \equiv X'(\omega(z))/\Delta t$, if we transcribe the convolution result $A'(\omega) = B'(\omega) C'(\omega)$ of (23.4) , we pick up an extra factor of Δt . Thus we compare convolution theorems :

$$a_n = \sum_{m=-\infty}^{\infty} \Delta t\ b_{n-m} c_m \qquad \Leftrightarrow \qquad A'(\omega) = B'(\omega)C'(\omega) \qquad \textit{(22.8)}$$

$$a_n = \sum_{m=-\infty}^{\infty} \Delta t\ b_{n-m} c_m \qquad \Leftrightarrow \qquad A''(z) = \Delta t\, B''(z)\, C''(z) \qquad (24.5)$$

Dimensions: Dim(a,c,A",C") = V, Dim(b,B") = $\sec^{-1}$.

In (24.5), it is convenient to absorb the Δt factor into the b_{n-m} coefficients. To do this, we define

$$h_n \equiv \Delta t\ b_n \qquad (24.6)$$

so that (24.5) may be written in this simpler and more traditional form, where H"(z) is the Z Transform of h_n,

$$a_n = \sum_{m=-\infty}^{\infty} h_{n-m} c_m \qquad \Leftrightarrow \qquad A''(z) = H''(z)\, C''(z) \ . \qquad (24.7)$$

Dimensions: Dim(a,c,A",C") = V, Dim(h,H") = 1.

Equation (24.7) is the digital convolution theorem stated in terms of the Z Transform.

(b) Unit Impulse

The analog of the unit impulse $\delta(t-a)$ at t=a must be a sequence of numbers x_n which are all zero except the one say at some integer m. We might write this as

$$x_n = \delta_m(n) \equiv \delta_{n,m} \qquad \text{"unit impulse"} \qquad n = \text{all integers, the sequence index} \qquad (24.8)$$

If we stuff this into the Z-Transform projection (24.3) , we get

$$X''(z) = z^{-m} \ . \qquad \text{" unit impulse response"} \qquad (24.9)$$

This looks a lot like our Fourier Integral result (8.2) (setting $t_1 = t_m$)

$x(t) = \delta(t - t_m)$

$$X(\omega) = e^{-i\omega t_m} = e^{-i\omega\Delta t m} = z^{-m} \quad . \qquad (8.2)$$

The expression z^{-m} on the right is the same in both cases. The Fourier Integral Transform does to its appropriate "unit impulse" just what the Z Transform does to *its* appropriate "unit impulse". The unit impulses are different.

In a filter with input I and output O we have O"(z) = H"(z) I"(z), where H"(z) is the filter transfer function. If I is taken to be a unit impulse at time t=0, then from the above I"(z) = z^0 = 1. Thus, quantity H"(z) *is* the z-domain response of the filter to an impulse at t=0. From (24.4) one can then get the time domain impulse response h_n. We shall do this below for an "RC" filter.

(c) Time Translation

Above we show a unit impulse $\delta_m(n)$ at time m and its Z transform z^{-m}. A unit impulse one step *later* in time would be $\delta_{m+1}(n)$, and its Z transform would be $z^{-m-1} = z^{-m}\, z^{-1}$. This suggests that if a signal is delayed by one time step, its Z transform acquires a factor z^{-1}. Advancing a signal one step means multiply by z^{+1}. These facts are true for an arbitrary signal; they follow immediately from (24.4):

$$x_{n+1} = \frac{1}{2\pi i}\int_C dz\, X''(z)\, z^{n-1}\, z^{+1} \qquad \text{// advance one step} \qquad x_{n+1} \leftrightarrow z^{+1}X''(z) \qquad (24.10)$$

$$x_{n-1} = \frac{1}{2\pi i}\int_C dz\, X''(z)\, z^{n-1}\, z^{-1} \qquad \text{// delay one step} \qquad x_{n-1} \leftrightarrow z^{-1}X''(z)\ . \qquad (24.11)$$

A very similar thing happens in the Fourier Integral Transform world, where time translation generates a multiplicative phase as shown in (12.1): $x(t - t_1) \leftrightarrow X(\omega)\, e^{-i\omega t_1}$.

In general, one can delay a digital signal one step in time by running it through a D flip-flop having clock period Δt, so this is why such flip-flops are associated with z^{-1} in a digital filter. There is no analogous device to associate with z^{+1}. It would have to be a causality-violating device.

(d) Derivative Limit

Based on the preceding subsection, we know that the following difference of two sequences has this transform,

$$\frac{x_n - x_{n-1}}{\Delta t} \quad \leftrightarrow \quad X''(z)\left[\frac{1 - z^{-1}}{\Delta t}\right] \ . \qquad (24.12)$$

This is just a simple superposition. If we take the limit $\Delta t \to 0$, the LHS becomes dx/dt, and the RHS becomes X"(z) iω, since $z^{-1} = \exp(-i\omega\Delta t) \approx 1 - i\omega\Delta t$. Since $\lim_{\Delta t\to 0} [\ \Delta t\ X''(z)] = X(\omega)$, we are not surprised to find that the iω rule (11.1) applies to both X"(z) and X(ω).

(e) Digital RC filter

This filter was treated in terms of the Fourier Transform in Section 4 (b) where we wrote its analog description in (4.5),

$$RC\ dv_o(t)/dt + v_o(t) = v_i(t)\ . \qquad \textit{(4.5)} \qquad (24.13)$$

Undoing the limit as just described above, we write this in digital form as

$$RC\ \frac{v_o(t_n) - v_o(t_{n-1})}{\Delta t} + v_o(t_n) = v_i(t_n)\ . \qquad (24.14)$$

For any finite Δt, this equation is of course different from (24.13) but for small Δt we expect it to be a good approximation for the system described by (24.13).

Letting $o_n \equiv v_o(t_n)$ and $i_n \equiv v_i(t_n)$ this reads

$$RC\ \frac{o_n - o_{n-1}}{\Delta t} + o_n = i_n \quad . \qquad (24.15)$$

Z Transform each of the four terms shown and use (24.11) on o_{n-1} to get

$$RC\ O''(z)\ [\frac{1 - z^{-1}}{\Delta t}] + O''(z) = I''(z)$$

or

$$[\ \alpha\ (1-z^{-1}) + 1]\ O''(z) = I''(z)\ . \qquad \alpha \equiv (RC/\Delta t) = \text{dimensionless}$$

If we want to interpret this circuit as a digital filter, we write, as in (24.7),

$$O''(z) = H''(z)\ I''(z)\ . \qquad (24.16)$$

The filter transfer function is then

$$H''(z) = \frac{1}{\alpha(1-z^{-1}) + 1} = \frac{z}{\alpha(z-1) + z} = \frac{z}{(\alpha+1)z - \alpha} = \frac{z/(1+\alpha)}{z - \alpha/(1+\alpha)} \quad . \qquad (24.17)$$

We may now use (24.4) to recover the time domain signal,

$$h_n = \frac{1}{2\pi i} \int_C dz\ H''(z)\ z^{n-1} = \frac{1}{2\pi i(1+\alpha)} \int_C dz\ \frac{z^n}{z - \alpha/(1+\alpha)} \quad . \qquad (24.18)$$

For $n \geq 0$, the integrand has a single pole at location $z = \alpha/(1+\alpha)$ where $\alpha = (RC/\Delta t)$. As α ranges from 0 to ∞, the pole location moves from 0 to 1, so it is always inside the unit circle contour,

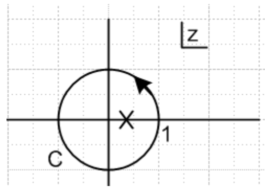


Fig 24.2

The integral may be evaluated as $2\pi i$ times the residue at this pole :

$$h_n = \frac{1}{2\pi i(1+\alpha)} \, 2\pi i \left(\frac{\alpha}{1+\alpha}\right)^n = \frac{1}{\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{n+1} = \frac{\Delta t}{RC}\left(\frac{\alpha}{1+\alpha}\right)^{n+1} .$$

The samples h_n are dimensionless, but to compare with our earlier RC work we write as in (24.6) that $h_n = \Delta t \, g_n$ to get

$$g_n = \frac{1}{RC}\left(\frac{\alpha}{1+\alpha}\right)^{n+1} . \tag{24.19}$$

In the case $n < 0$, in addition to the pole just mentioned, there is a pole or order n at $z = 0$. But when $n < 0$, we can expand the contour out to a Great Circle at infinity and the $z^{-|n|}$ factor then causes the integral to vanish. The reason is that in this limit we have, with $z = Re^{i\theta}$,

$$\int_{GC} dz \; z^{-|n|-1} = \int_0^{2\pi} Ri\, e^{i\theta} \; e^{-i\theta(|n|+1)} \, R^{-(|n|+1)} = R^{-|n|} \left\{ \int_0^{2\pi} e^{-i\theta|n|} \right\} . \tag{24.20}$$

But the integral {..} is finite, and as $R\to\infty$, $R^{-|n|} \to 0$ for n = -1,-2... so the GC integral is 0.

Thus, in terms of the Heaviside Step function,

$$g_n = g(t_n) = (1/RC) \; (1+\alpha^{-1})^{-n-1} \; \theta(n+\varepsilon) \qquad\qquad \alpha = (RC/\Delta t) \tag{24.21}$$

where $\varepsilon > 0$ is any quantity less than 1 so we avoid the fact that $\theta(0) = 1/2$. We can compare this to the analog output of the true RC filter (4.10)

$$g(t) = (1/RC)\, e^{-(t/RC)} \; \theta(t) \quad . \tag{4.10}$$

Both the analog and digital filters demonstrate causality with the θ factors shown. However, the analog filter decays in an exponential fashion, whereas the digital decays in a geometric manner. We can rewrite the digital result in this manner

$$g_n = g(t_n) = (1/RC)\, e^{-(n+1)\ln(1+1/\alpha)} \; \theta(n+\varepsilon) \; . \tag{24.22}$$

In the limit $\Delta t \ll RC$ we have $\alpha \ll 1$ and then $\ln(1+1/\alpha) \approx 1/\alpha = \Delta t/(RC)$ and this result becomes

$$g(t_n) = (1/RC)\, e^{-(n+1)\Delta t/(RC)}\ \theta(n+\varepsilon) = (1/RC)\, e^{-t_{n+1}/(RC)}\ \theta(n+\varepsilon) \qquad (24.23)$$

and, since $t_{n+1} = t_n + \Delta t \approx t_n$, this replicates the analog result (4.10) in the small Δt limit.

So we learn that, in order to make our digital RC filter produce the same results as an analog RC filter, we must take $\Delta t << RC$, which is no big surprise since this was assumed at the start going from (24.13) to the difference equation (24.14).

(f) Poles in H"(z) imply feedback and infinite impulse response (IIR)

A general form for an implementable dimensionless transfer function H"(z) is a ratio of polynomials in z. We saw an example in the RC filter (24.17) above. So consider this general form,

$$H''(z) = [\Sigma_{n=0}^{M} a_n z^n] / [\Sigma_{n=0}^{N} b_n z^n] . \qquad (24.24)$$

In the *special case* that the denominator has the form $\Sigma_{n=1}^{N} b_n z^n = z^k$ for some integer $k \geq M$, we have

$$H''(z) = [\Sigma_{n=0}^{M} a_n z^n] / (z^k) = \Sigma_{n=0}^{M} a_n\, z^{n-k} = \Sigma_{n=0}^{M} a_n\, (z^{-1})^{k-n} .$$

Since $k \geq M$, the exponents on $(z^{-1})^{k-n}$ are all non-negative. In this case we can write

$$H''(z) = a_0(z^{-1})^k + a_1(z^{-1})^{k-1} + \ ..... \ + a_M (z^{-1})^{k-M}$$

$$= a_0 z^{-k} + a_1 z^{-k+1} + \ ..... \ + a_M\, z^{-k+M} \qquad k \geq M \qquad (24.25)$$

where all the (z^{-1}) exponents are non-negative integers. As we shall show by example below, since a filter transfer function having this general form has only positive powers of (z^{-1}), it can be implemented by hardware which has no feedback loops, and which therefore has an output which dies out some finite number of clocks after the input dies out. If this filter is given an impulse as input, the output dies out after a certain number of clocks. Thus, the filter has a finite impulse response and is then called a **Finite Impulse Response** or **FIR** filter (example below).

Notice that in our special case H"(z) has a pole of order k at z = 0. When we later claim that transfer functions having poles must be implemented in hardware with feedback giving an infinite impulse response, we are referring to poles not located at z = 0.

For our example we shall assume k = M = 2. Then,

$$H''(z) = a_0 z^{-2} + a_1 z^{-1} + a_2 = A + Bz^{-1} + Cz^{-2}. \qquad (24.26)$$

Then if I"(z) and O"(z) are the input and output of our filter,

$$O''(z) = H''(z)\, I''(z), \qquad \textit{(24.16)}$$

we have

$$O''(z) \;=\; [A + Bz^{-1} + Cz^{-2}]\, I''(z) \;= A\, I''(z) + Bz^{-1}\, I''(z) + Cz^{-2}\, I''(z) \qquad (24.27)$$

Using the shift rule (24.11) we can translate the above equation into the time domain to get

$$o_n \;=\; A\, i_n + B\, i_{n-1} + C\, i_{n-2}\,. \qquad (24.28)$$

These last two equations can be represented by these diagrams:

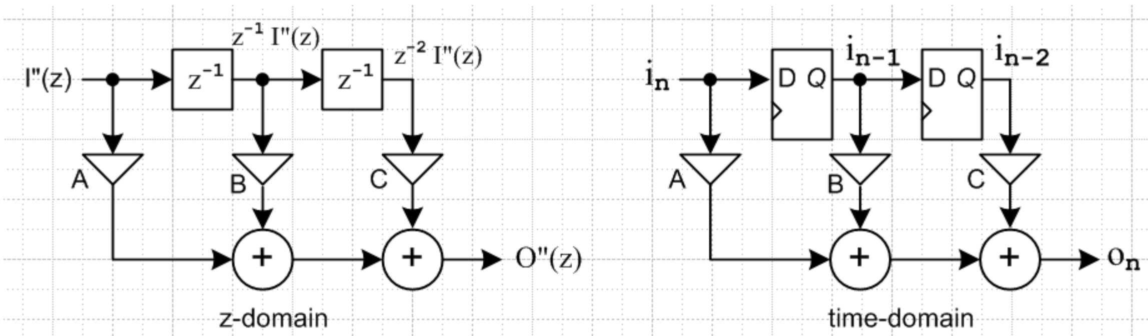
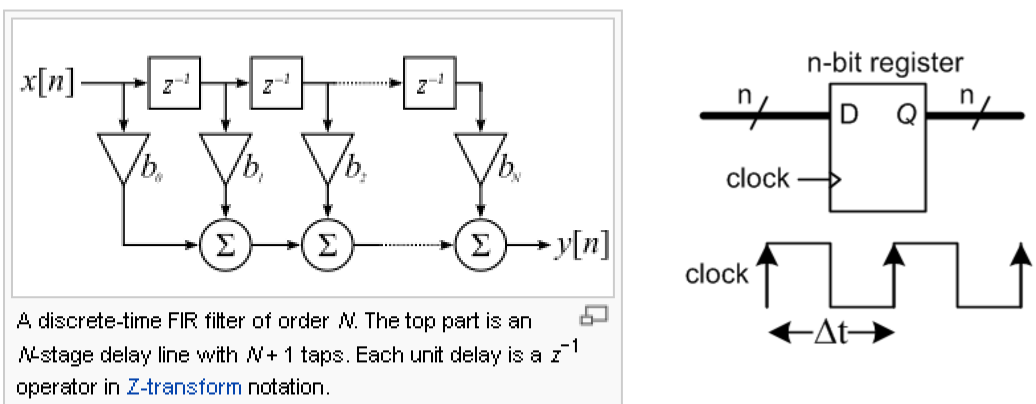


Fig 24.3

The time-domain diagram represents a piece of "hardware" wherein the output is developed with two D flip-flop registers (clock period Δt) and three constant multipliers and two adders. This hardware circuit has no "feedback" because no flip-flop output is ever involved in determining a flip-flop input. Sometimes authors combine these two pictures, drawing the time domain register elements as boxes with z^{-1} labels inside. This convention appears in the following wiki picture (left),



http://en.wikipedia.org/wiki/Finite_impulse_response

Fig 24.4

On the right we show the usual notation for attaching a clock to a register. In these pictures, each line (but not the clock) represents a "bus" of however many bits n is used to represent a digital sample. The triangle symbol for a multiplier suggests an "amplifier" which scales a signal. Later we shall use a simple X placed on a bus to indicate multiplication by a constant. The flip-flops are clocked by a square-wave clock pulse train having period Δt. At each positive edge of the clock signal, the value which the register input has just before that edge is loaded into the register. The register output then holds that value constant until the next positive clock edge.

Here is the response of the circuit of Fig 24.3 to a unit impulse aligned with i_1 :

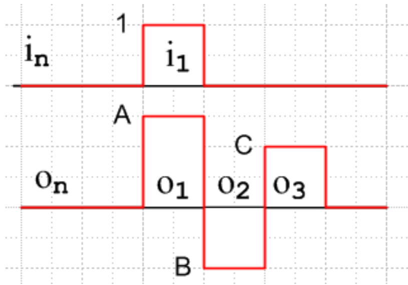


Fig 24.5

and it seems pretty clear that the impulse response is finite.

We now consider a different example with poles. Suppose H"(z) is the inverse of that of the previous example,

$$H''(z) = \frac{z^2}{Az^2+Bz+C} = \frac{1}{A+Bz^{-1}+Cz^{-2}} \qquad (24.29)$$

so now H"(z) has some non-zero poles (poles not at z=0). Then we get I and O swapped, so

$$I''(z) = [\ A + Bz^{-1} + Cz^{-2}\ ]\ O''(z)\ . \qquad (24.30)$$

Solve this for O"(z) in the following manner (solve for A O"(z) then divide by A),

$$O''(z) = (1/A)\ I''(z) + (-\ B/A)\ z^{-1}\ O''(z) + (-\ C/A)\ z^{-2}\ O''(z)\ . \qquad (24.31)$$

Translating this to the time domain using rule (24.11) gives,

$$o_n = (1/A)\ i_n\ +\ (-\ B/A)\ o_{n-1}\ +\ (-\ C/A)\ o_{n-2}\ . \qquad (24.32)$$

The corresponding drawings are these :

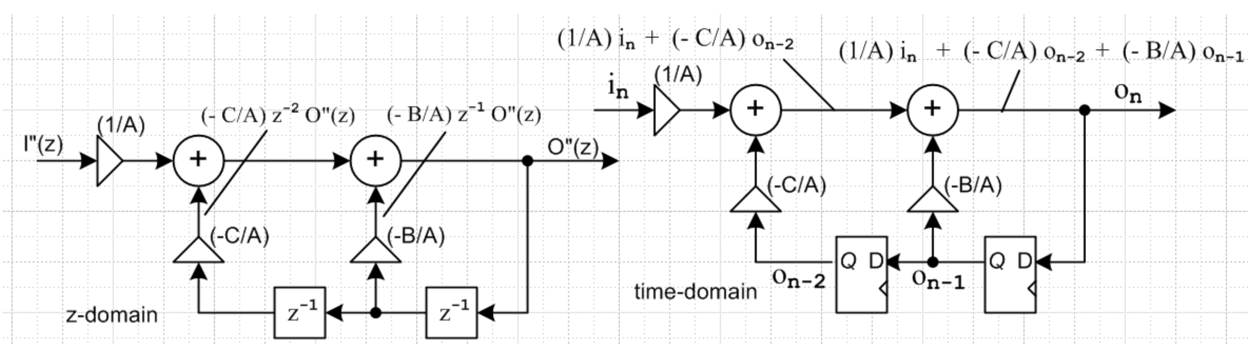


Fig 24.6

Here one can see the feedback: register inputs are dependent on register outputs. This is an **Infinite Impulse Response** (**IIR**) filter, since the impulse response h_n carries on forever due to the feedback. See

h_n in (24.19) as another example: geometric decay which never goes away completely. In that example the transfer function H(z") has a pole as shown in (24.17). [The response might go away in a real digital circuit with a finite number of quantization bits.]

(g) The Digital RC filter revisited

We can now draw up the RC filter discussed above. We had in (24.16) and (24.17),

$$O''(z) = [H''(z)]I''(z) = \frac{z/(1+\alpha)}{z - \alpha/(1+\alpha)} I''(z) = \frac{1/(1+\alpha)}{1 - z^{-1}\alpha/(1+\alpha)} I''(z) \qquad (24.33)$$

or

$$O''(z) - \frac{\alpha}{1+\alpha} z^{-1} O''(z) = \frac{1}{1+\alpha} I''(z)$$

or

$$O''(z) = \frac{\alpha}{1+\alpha} z^{-1} O''(z) + \frac{1}{1+\alpha} I''(z) \qquad (24.34)$$

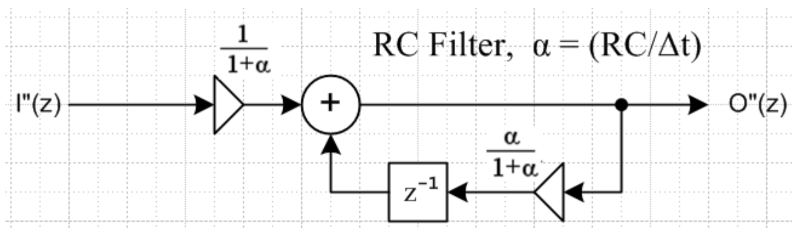


Fig 24.7

As just noted above, the presence of a non-zero pole in H"(z) gives feedback.

Defining $\beta \equiv 1/\alpha = (\Delta t/RC)$, then in the regime in which the filter is accurate $\alpha >> 1$ so $\beta << 1$, and then

$$\frac{1}{1+\alpha} = \frac{1/\alpha}{1+1/\alpha} = \frac{\beta}{1+\beta} \approx \beta(1-\beta) \approx \beta$$

$$\frac{\alpha}{1+\alpha} = \frac{1}{1+1/\alpha} = \frac{1}{1+\beta} \approx (1-\beta) \approx e^{-\beta} , \qquad (24.35)$$

which yields another form which sometimes appears in textbooks (Lam pages 509 and 503)

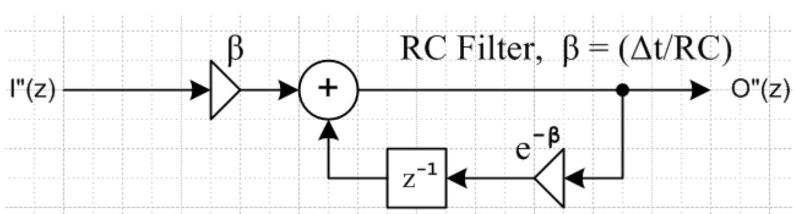


Fig 24.8

If we use *these* constants in (24.17) we get the rational polynomial transfer function

$$H''(z) = \frac{z\beta}{z - e^{-\beta}}$$

and then from (24.18) with $h_n = \Delta t\, g(t_n)$,

$$g(t_n) = \frac{1}{RC}\, e^{-n\Delta t/RC} = \frac{1}{RC}\, e^{-t_n/RC}\, \theta(n+\varepsilon) \quad . \tag{24.36}$$

This is an accurate result even when α is not large (β not small), so although this is not the design that emerged from our small-Δt analysis in section (e) above, it is certainly a better design for a digital RC filter. The two designs produce the same output for Δt << RC.

(h) Other circuits

Typical examples of IIR filters having feedback are serial scramblers and CRC generators. One thinks of the input sequence I"(z) as a huge polynomial in z, where the presence or absence of each power represents a 1 or a 0. That is, think of the incoming stream as a superposition of unit impulses with weights equal to the binary digits of the data stream. The transfer function H"(z) = 1/polynomial, so we write O"(z) = H"(z) I"(z) = I"(z)/polynomial. The output stream is then the quotient of polynomial division. One way to interpret the above discussion is as follows:

no poles → polynomial multiplication → no feedback → FIR

poles with no zeros → polynomial division → feedback → IIR

poles and zeros → simultaneous polynomial multiplication and division → feedback → IIR

This subject will be pursued more in a separate document.

(Z Transform summary box on next page)

(i) Z Transform Summary

Z Transform (24.37)

1. Let x(t) be any reasonable function.

2. Divide up the time axis into steps $t_n = n\,\Delta t$; let $T_1 = \Delta t$ and $\omega_1 = 2\pi/T_1$.

3. We can think of samples $x_n = x(t_n)$ for the above x(t). Alternatively, we can think of the x_n as some given sequence, and one could then construct an infinite number of functions x(t) for which $x_n = x(t_n)$.

4. In terms of x(t), the Z Transform and its inverse are given by

$$X''(z) = \sum_{n=-\infty}^{\infty} x(t_n)\, z^{-n} \qquad (24.3)$$

$$x(t_n) = \frac{1}{2\pi i} \int_C dz\, X''(z)\, z^{n-1} \ . \qquad (24.4)$$

More generally, dispensing now with x(t) and using just x_n,

$$X''(z) = \sum_{n=-\infty}^{\infty} x_n\, z^{-n} \qquad (24.3)$$

$$x_n = \frac{1}{2\pi i} \int_C dz\, X''(z)\, z^{n-1} \qquad (24.4)$$

The contour C goes once counterclockwise around the unit circle in the z-plane.

5. The Z Transform diagonalizes any convolution sum :

$$a_n = \sum_{m=-\infty}^{\infty} \Delta t\ b_{n-m}\, c_m \quad \Leftrightarrow \quad A''(z) = \Delta t\, B''(z)\, C''(z) \qquad (24.5)$$

$$a_n = \sum_{m=-\infty}^{\infty} h_{n-m}\, c_m \quad \Leftrightarrow \quad A''(z) = H''(z)\, C''(z) \qquad (24.7)$$

where $h_n \equiv \Delta t\ b_n$ and $H''(z) = \Delta t\, B''(z)$

6. The Z transform is related to the Digital Fourier Transform of box (23.5) by

$$X''(z) \equiv \frac{1}{\Delta t}\, X'(\omega) \qquad \text{where} \quad z = e^{i\omega\Delta t} \qquad (24.2)$$

25. Amplitude Modulated Pulse Trains

In Section 14 we studied the spectrum of a simple pulse train made by superposing equal pulses $x_{pulse}(t)$ with spacing T_1. We found that the Fourier Integral spectrum $X(\omega)$ of such a pulse train was given by an infinite sequence of delta function spikes with amplitudes determined by an envelope function $c(\omega)$ which is just a multiple of the spectrum of the pulse (summary box 14.12).

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - t_n) \qquad t_n = nT_1 \qquad \text{pulse train} \qquad (14.1)$$

$$X(\omega) = \sum_{m=-\infty}^{\infty} c(\omega)\ 2\pi\ \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} c_m\ 2\pi\ \delta(\omega - m\omega_1) \qquad \text{spectrum} \qquad (14.9)$$

$$\text{where } c(\omega) \equiv (1/T_1)X_{pulse}(\omega) = (1/T_1) \int_{-\infty}^{\infty} dt\ x_{pulse}(t)\ e^{-i\omega t} \qquad (14.8) \text{ and } (1.1)$$

The numbers $c_m = c(m\omega_1)$ turned out to be exactly the complex Fourier Series coefficients.

Later in Section 20 we studied an amplitude-modulated pulse train in which $x_{pulse}(t) = T_1\delta(t)$, and we took note of the continuous spectrum of such a pulse train,

$$x(t) = y(t)\ d(t) = \sum_{n=-\infty}^{\infty} y_n\ T_1\delta(t - nT_1) \qquad y_n = y(nT_1) \qquad (20.3)$$

$$X(\omega) = Y'(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) = Y(\omega) + \sum_{m \neq 0} Y(\omega - m\omega_1) \quad . \qquad (20.7)$$

where $Y'(\omega)$ was the Digital Fourier Transform of $y(t)$ shown in items 3 and 4 of box (25.3).

In this section we shall combine both these ideas to obtain an amplitude modulated pulse train with an arbitrary pulse shape $x_{pulse}(t)$. We continue to denote the amplitude modulated pulse train by $x(t)$,

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\, x_{pulse}(t - t_n). \qquad (25.1)$$

What is the spectrum of this new pulse train? To find out, we insert (25.1) into (1.1),

$$X(\omega) = \int_{-\infty}^{\infty} dt\ x(t)\ e^{-i\omega t} = \int_{-\infty}^{\infty} dt\ [\sum_{n=-\infty}^{\infty} y_n\, x_{pulse}(t - t_n)]\ e^{-i\omega t}$$

$$= \sum_{n=-\infty}^{\infty} y_n \int_{-\infty}^{\infty} dt\ x_{pulse}(t - t_n)]\ e^{-i\omega t} = \sum_{n=-\infty}^{\infty} y_n\ [\int_{-\infty}^{\infty} dt'\ x_{pulse}(t')\ e^{-i\omega t'}]\ e^{+i\omega t_n} \qquad // \ t' = t - t_n$$

$$= \sum_{n=-\infty}^{\infty} y_n\, X_{pulse}(\omega)\ e^{+i\omega t_n} = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} y_n\, e^{+i\omega t_n}$$

where we used (1.1) to recognize $X_{pulse}(\omega)$. Recall now the Digital Fourier Transform as summarized in the box (23.5). From item 4 in that box, we can interpret the n sum above in this way ($\Delta t = T_1$)

$$\sum_{n=-\infty}^{\infty} y_n e^{+i\omega t_n} = \frac{1}{T_1} Y'(\omega) \quad = Y''(z) \tag{25.2}$$

which says, apart from a constant factor, this sum is the Digital Fourier Transform of y(t). [In this and the following equations, we will try to show results in terms of both the Digital Fourier Transform $Y'(\omega)$ and the Z Transform $Y''(z) = Y'(\omega)/T_1$.]

From item 6 in that same box, we know that $Y'(\omega) = [\ Y(\omega) + \sum_{m\neq 0} Y(\omega - m\omega_1)\ ]$. We conclude the above calculation of the spectrum $X(\omega)$ to find that, using the definition (14.8) of $c(\omega)$,

$$X(\omega) = X_{pulse}(\omega)\frac{1}{T_1} Y'(\omega) \quad = c(\omega)\, Y'(\omega) \ = X_{pulse}(\omega)\, Y''(z) \ . \tag{25.3}$$

Comparing this to (20.7) quoted just above, we see that the spectral effect of replacing the $T_1\delta(t)$ pulse by $x_{pulse}(t)$ is the addition of the pulse spectrum $c(\omega) = X_{pulse}(\omega)/T_1$ as an overall factor. We then recover the delta function result as a special case where $c(\omega) = T_1/T_1 = 1$ as at the start of Section 20.

Thus we arrive at this very significant result which deserves its own box:

Amplitude Modulated Pulse Train (25.4)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\, x_{pulse}(t - t_n) \qquad \textit{(25.1)}$$

$$X(\omega) = c(\omega)\, Y'(\omega) = c(\omega)\,[\, Y(\omega) + \sum_{m\neq 0} Y(\omega - m\omega_1)] = X_{pulse}(\omega)\, Y''(z) \qquad \textit{(25.3)}$$

$$c(\omega) = (1/T_1)X_{pulse}(\omega) = (1/T_1)\int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-i\omega t} \qquad \textit{(14.8) and (1.1)}$$

$$Y'(\omega) = T_1 Y''(z) = T_1 \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} \qquad \text{projection = transform} \qquad \textit{(23.5)}$$

The boxed result above is one of the holy grails of the spectral analysis of digital signals. Notice that by selecting y_n to vanish outside some range, the box applies to both infinite and finite pulse trains.

Example 1: A finite pulse train

Consider this finite sequence of y_n samples

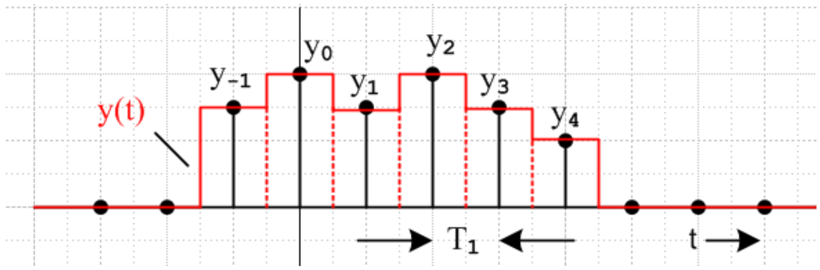


Fig 25.1

We can regard the red outline curve as an amplitude modulated pulse train whose pulse shape is a box of height A = 1 and width $\tau = T_1$. The Fourier Integral Transform spectrum of this box from (9.2) is

$$X_{pulse}(\omega) = T_1 \,\mathrm{sinc}(\omega T_1/2). \tag{9.2}$$

From box (25.4) the Fourier Integral Transform spectrum X(ω) of the pulse train is given by

$$X(\omega) = (1/T_1)X_{pulse}(\omega) \;\{ Y'(\omega) \} = X_{pulse}(\omega) \,[Y'(\omega)/T_1]$$

$$= \mathrm{sinc}(\omega T_1/2) \{ Y'(\omega) \} = \mathrm{sinc}(\omega T_1/2) \;\{ T_1 \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \} \tag{25.5}$$

where Y'(ω) is the Digital Fourier Transform of the sequence y_n as shown in (23.5).

Here is a Maple plot of | Y'(ω) | with $T_1 = 1$.

```
y := array(-1..4,[3,4,3,4,3,2]):
Yp := w -> T1*sum(y[n]*exp(-I*n*w*T1),n=-1..4):
plot(abs(Yp(w)), w = -20..20, numpoints = 600);
```

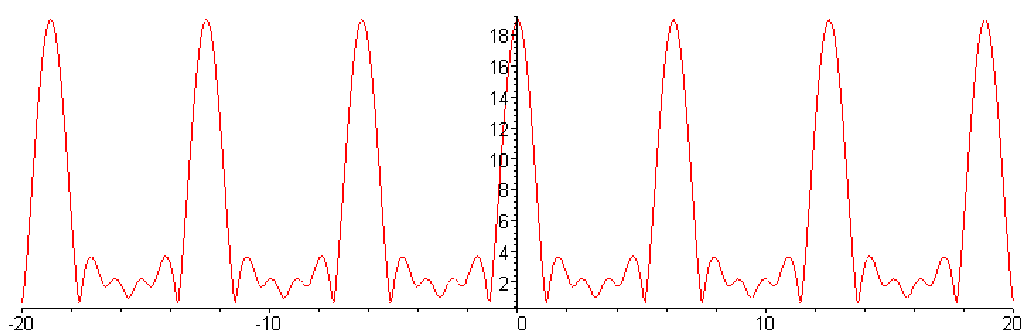


Fig 25.2

where we see the expected image spectra at $N\omega_1 = N2\pi$. There is a strong DC component at ω = 0 and the peak there is the sum 19 of the y_n values (all of which are positive) so X(0) = Y'(0) = 19 .

Next we show in red a plot of | X(ω) | from (25.5) for the finite pulse train,

```
sinc := x -> sin(x)/(x):
X := w -> sinc(w*T1/2)*Yp(w):
plot([abs(X(w)),abs(19*sinc(w*T1/2))], w = -20..20, color = [red,blue],numpoints =300);
```

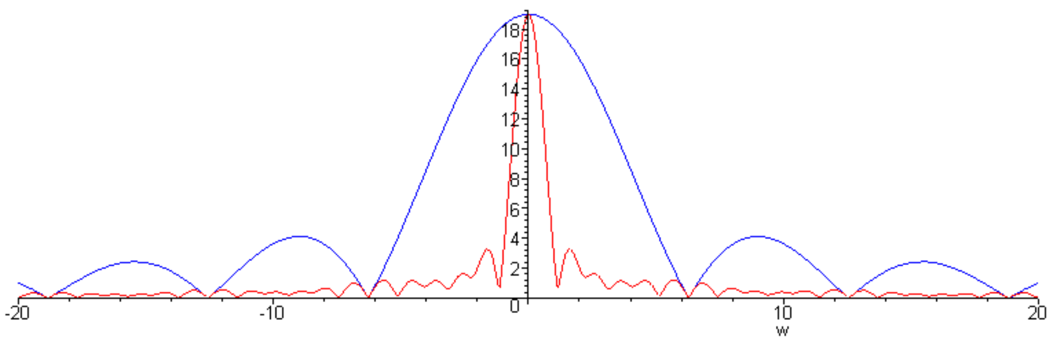


Fig 25.3

where the blue curve provides an outline of X(0) |sinc($\omega T_1/2$)| . The red curve is the spectrum of the physical red analog signal shown in Fig 25.1 and there are no image spectra. The sinc function in this example crushes out the image spectra with its zeros.

Now we shall attempt some reconstructions.

First, we reconstruct the y_n from Y'(ω) using the inversion formula in box (23.5),

$$y_k = \frac{1}{2\pi}\int_{-\omega_1/2}^{\omega_1/2} d\omega\, Y'(\omega)\, e^{+i\omega k T_1} \qquad \text{expansion = inversion} \qquad (23.5)$$

```
for k from -4 to 8 do
  yrecon[k] := (1/(2*Pi))*int(Yp(w)*exp(I*w*k*T1), w = -w1/2..w1/2);
od:
print(seq(yrecon[k],k=-4..8));
```

0, 0, 0, 3, 4, 3, 4, 3, 2, 0, 0, 0, 0

which are in fact the y_n we started with.

Second, we reconstruct the pulse train x(t) from X(ω) using the inversion formula (1.2),

$$x(t) = (1/2\pi)\int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

```
X := w -> sinc(w*T1/2)*Yp(w):
s := (1/(2*Pi))*X(w)*exp(I*w*t):
f := t -> int(Re(s),w=-infinity..infinity):
f(t);
```

$$\frac{1}{2}\mathrm{signum}\left(-\frac{3}{2}+t\right)+\frac{3}{2}\mathrm{signum}\left(\frac{3}{2}+t\right)+\frac{1}{2}\mathrm{signum}\left(\frac{1}{2}+t\right)-\frac{1}{2}\mathrm{signum}\left(-\frac{1}{2}+t\right)-\frac{1}{2}\mathrm{signum}\left(-\frac{5}{2}+t\right)-\frac{1}{2}\mathrm{signum}\left(-\frac{7}{2}+t\right)-\mathrm{signum}\left(-\frac{9}{2}+t\right)$$

```
plot(f(t),t=-4..8, thickness=3,xtickmarks =[seq(n,n=-4..8)],numpoints=100);
```

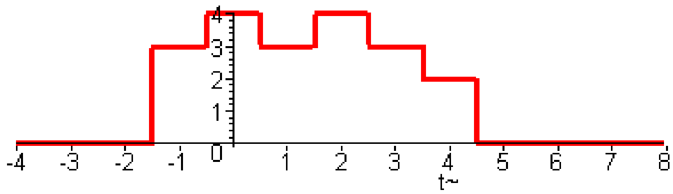


Fig 25.4

which replicates our starting figure. The reason for Re(s) is that that s has a tiny imaginary part ~ 10^{-9} due to calculational error, and the plot routine requires a real function. Maple does the integral analytically as shown (see (C.14) for signum).

Example 2: The unit impulse and the sinc sum rule

Even the simplest case is interesting. The y_n sequence is taken as a unit impulse scaled by y_0

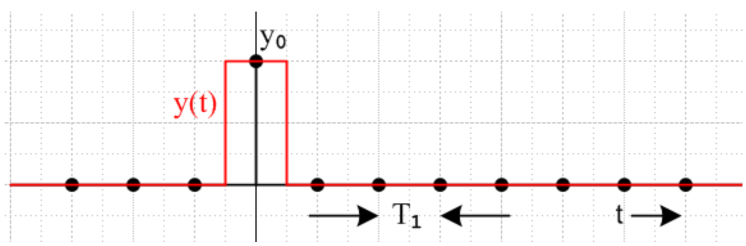


Fig 25.5

$X_{pulse}(\omega) = T_1\ \mathrm{sinc}(\omega T_1/2)$. // for box of unit height *(9.2)*

From box (25.4) the Fourier Integral Transform spectrum X(ω) of this "pulse train" is given by

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\ \{ Y'(\omega) \}$$
$$= \mathrm{sinc}(\omega T_1/2)\ \{ T_1 \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \}$$
$$= \mathrm{sinc}(\omega T_1/2)\ \{ T_1 y_0 \} = \mathrm{sinc}[\pi(\omega/\omega_1)]\ \{ T_1 y_0 \} \quad // = Y(\omega) \qquad (25.6)$$

so in this example $Y'(\omega) = T_1 y_0$. If we regard the red plot as y(t), then X(ω) = Y(ω), the Fourier Integral Transform of y(t). They are the same since this pulse train has only one pulse.

A plot of |X(ω)| = |Y(ω)| has a familiar look (T_1= 1, ω_1= 2π, y_0 = 1) :

```
Y := w -> sinc(w*T1/2)*T1*y0:
plot(abs(Y(w)),w=-40..40);
```

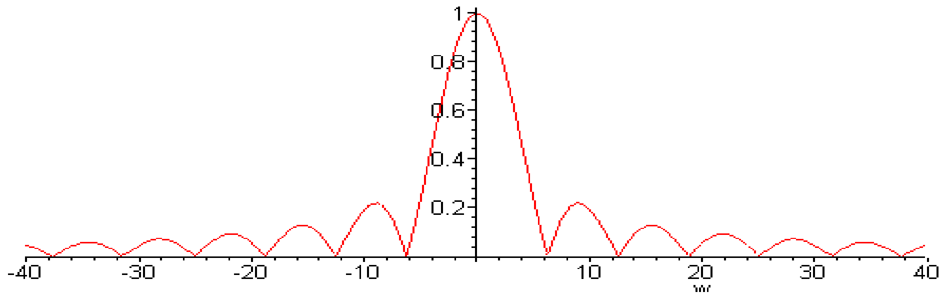


Fig 25.6

We have just noted that $Y'(\omega) \equiv y_0T_1 = 1$. How exactly does the image spectrum equation in box (23.5) item 6 , namely

$$Y'(\omega) = [\ Y(\omega) \ + \Sigma_{m\neq 0}Y(\omega - m\omega_1)\ ]\ , \tag{25.7}$$

work out? First, we plot the right side of (25.7) limiting the sum range to m = -100 to 100 :

```
Yp_approx := w -> sum(Y(w-m*w1),m=-100..100):
plot(Re(Yp_approx(w)),w=-1000..1000);
```

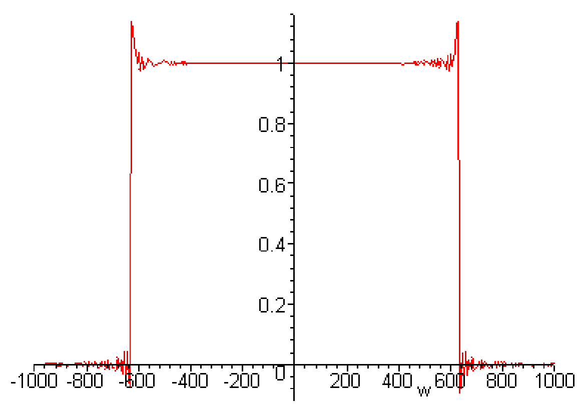


Fig 25.7

It appears that the shifted sinc functions are adding up to produce the constant $Y'(\omega) = 1$. In terms of the math, this must mean that

$$Y'(\omega) = \sum_{m=-\infty}^{\infty} Y(\omega - m\omega_1) \ = y_o\, T_1 \sum_{m=-\infty}^{\infty} \text{sinc}[\pi(\omega/\omega_1 - m)] \ = y_oT_1 \tag{25.8}$$

which implies the following unusual sum rule, valid for any real x :

$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x\text{-}m)] = 1\ . \tag{25.9}$$

This result is sometimes quoted with x = 0 but it is in fact valid for any x. We have in effect proven the sum rule with the above analysis, but as usual we would like to find verification. First process the sum as follows,

$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x-m)] = \sum_{m=-\infty}^{\infty} \frac{\sin[\pi(x-m)]}{\pi(x-m)} = (1/\pi)\sin[\pi x] \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{x-m} \quad .$$

The sum on the right can be further processed,

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{x-m} = \frac{1}{x} + \left[\sum_{m=-\infty}^{-1} + \sum_{m=1}^{\infty} \right] \frac{(-1)^m}{x-m} = \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{(-1)^m}{x^2-m^2} \ .$$

According to Gradshteyn and Ryzhik 142.3 (page 44),

3. $\operatorname{cosec} \pi x = \dfrac{1}{\pi x} + \dfrac{2x}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2}$ (see also **1.217** 2) AD (6495.4)a

we may replace

$$\frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{(-1)^m}{x^2-m^2} = \pi \csc(\pi x)$$

and then we find that

$$\sum_{m=-\infty}^{\infty} \text{sinc}[\pi(x-m)] = (1/\pi)\sin[\pi x] \ \pi\csc(\pi x) = 1$$

which then verifies the sum rule (25.9).

26. A simple application: Aperture Correction

The output of a digital system usually involves a D/A converter followed by an analog post-filter. Let us assume that this system is attempting to reproduce some reasonable analog waveform y(t). Assume that each converted value is held as charge on a capacitor for some portion τ of the conversion period T_1, and then the capacitor charge is instantly dumped to ground for the remainder of the period. Period τ is called the aperture. In this way, we produce a signal x(t) that is a sequence of square pulses modulated by the values $y(t_n)$:

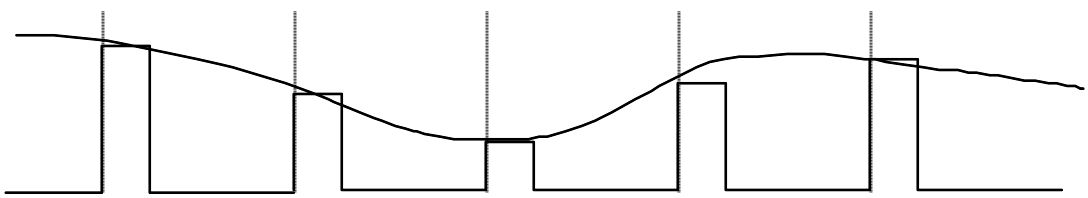


Figure 26.1. The smooth curve is y(t), the pulse train is x(t). Spacing is $\Delta t = T_1$. Width of each pulse is τ (the aperture), so duty cycle is τ/T_1. Fig 26.1

What is the spectrum of x(t)? It is an amplitude modulated pulse train, so according to (25.3) the spectrum of x(t) is

$$X(\omega) = (1/T_1)\, X_{pulse}(\omega)\, Y'(\omega) \ . \qquad (26.1)$$

The pulse $x_{pulse}(t)$ is a square pulse of unit height, width τ, and with its left edge aligned with t=0. We know the spectrum of this pulse from (9.2), but by (12.1) we must add a time-shift phase $\exp(-i\omega\tau/2)$ because we are translating our earlier pulse $\tau/2$ units to the right to make the left edge line up at t=0. Thus, from (9.2) and (12.1),

$$X_{pulse}(\omega) = \tau\, \mathrm{sinc}(\omega\tau/2)\, e^{-i\omega\tau/2} \qquad (26.2)$$

so the spectrum of x(t) is

$$X(\omega) = (\tau/T_1)\, \mathrm{sinc}(\omega\tau/2)\, \exp(-i\omega\tau/2)\, Y'(\omega) \ . \qquad (26.3)$$

The magnitude of $X(\omega)$ is the product of two functions which we now illustrate, where the grey humps represent $Y(\omega)$ and its images which combine to make $Y'(\omega)$, whatever it might be,

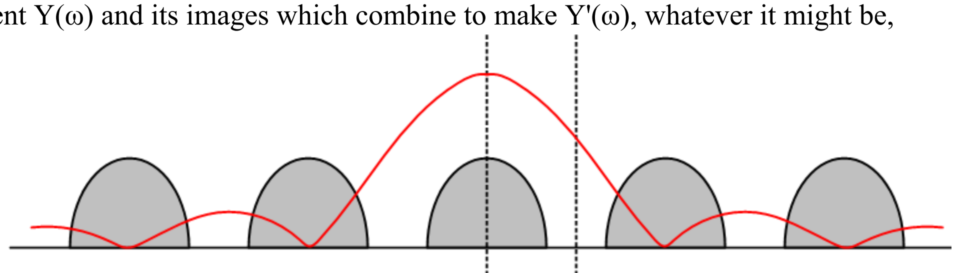


Figure 26.2. The humps are $|Y'(\omega)|$ and the red curve is $(\tau/T_1)\, |\mathrm{sinc}(\omega\tau/2)|$. Fig 26.2

Presumably our low pass analog post-filter is going to select out the portion of the spectrum indicated by the dotted lines in Figure 26.2. In this region we have,

$$X(\omega) = (\tau/T_1)\ \mathrm{sinc}(\omega\tau/2)\ \exp(-i\omega\tau/2)\ Y(\omega), \tag{26.4}$$

where $Y(\omega)$ is the main spectrum of $Y'(\omega)$. The factor $(\tau/T_1)\mathrm{sinc}(\omega\tau/2)$ represents an undesired magnitude distortion of the spectrum $X(\omega)$ due to the aperture τ. The phase $\varphi(\omega) = -i\omega\tau/2$ is a harmless linear phase which just means the whole signal is delayed by time $\tau/2$, as shown in Section 21 (b).

The distortion is at its *worst* when the aperture τ fills the entire period T_1, in which case the signal in Figure 26.1 looks like a traditional stepwise fit to y(t). The distortion is worst because the zeros of $\mathrm{sinc}(\omega\tau/2)$ are at $\omega_m = m\ (2\pi/\tau)$, so they are moved in as close as possible when τ is as large as possible, $\tau = T_1$.

The distortion can be reduced by making τ as small as practicable. In this case, the zeros move out, and the central hump of $\mathrm{sinc}(\omega\tau/2)$ is broad, so its drop-off during $Y(\omega)$ is minimized. Of course the amplitude (τ/T_1) of $X(\omega)$ also drops off as τ is made small, so there is a tradeoff.

In any event, there is still *some* distortion represented by $\mathrm{sinc}(\omega\tau/2)$ varying in the dotted region in Figure 26.2. Usually one attempts to correct for this aperture distortion by building into the analog post-filter an exactly compensating boost at frequencies near the cutoff region of the filter.

Thus, if the post-filter would normally be some $F(\omega)$ cutting off in the region of the second dotted line in Figure 26.2, a correcting filter would have the spectrum $F(\omega)/\mathrm{sinc}(\omega\tau/2)$. This filter needs to know the aperture time τ in addition to the cutoff frequency. Such a filter is said to have "sine x over x correction".

27. The Discrete Fourier Transform

Up to this point, $x_{pulse}(t)$ has always been considered a continuous function of time t. A simple pulse train was formed as $x(t) = \Sigma_n x_{pulse}(t-t_n)$ and an amplitude modulated pulse train as $\Sigma_n y_n x_{pulse}(t-t_n)$ where $t_n = nT_1$. Now for the first time we wish to consider a digital approximation to $x_{pulse}(t)$. That is the main subject of this section, and we shall develop it in analogy to Section 22 for the Digital Fourier Transform.

In a slight reversal of our normal order of doing things, in section (a) we shall develop the Discrete Fourier Transform (DFT) for a pulse train, then in section (b) we develop the Discrete Fourier Transform for an isolated pulse, this latter being the traditional form of the DFT.

(a) The Discrete Fourier Transform for a Simple Pulse Train x(t)

Recall from Section 15 the discussion of the Fourier Series Transform with complex coefficients c_m. This was summarized in box (15.12) from which we quote,

<u>Fourier Series Transform:</u> (complex form)

$$c_m \equiv (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-im\omega_1 t} = (1/T_1) \int_0^{T_1} dt\, x(t)\, e^{-im\omega_1 t} \qquad \textit{(14.16)} \qquad (27.1)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) = \sum_{m=-\infty}^{\infty} c_m\, e^{+im\omega_1 t} \qquad \textit{(14.1) + (15.9)} \qquad (27.2)$$

Here, x(t) is an infinite pulse train created by superposing pulses $x_{pulse}(t)$ at spacing T_1. Thus, x(t) is a periodic function with period T_1.

We wish now to redefine our concept of interval Δt. In our previous discussion, we set $\Delta t = T_1$. Here we wish instead to break up *each* interval T_1 into N pieces of size Δt, so now we have:

$$\begin{array}{ll} \Delta t = T_1/N & \omega_1 = (2\pi/T_1) = 2\pi/(N\Delta t) \\ T_1 = N\,\Delta t & (2\pi/N) = \omega_1 \Delta t \\ t_n = n\Delta t & \omega_1 t_n = n\,\omega_1\Delta t = n\,(2\pi/N)\,. \end{array} \qquad (27.3)$$

Here $t_n = n\Delta t$ represents a sequence of sample times for x(t). We still have our same periodic pulse train as in (14.1) which we then evaluate at discrete times $t = t_n$ to get (27.5).

$$x(t) = \sum_{m=-\infty}^{\infty} x_{pulse}(t-mT_1)\ . \qquad \textit{(14.1)} \qquad (27.4)$$

$$x(t_n) = \sum_{m=-\infty}^{\infty} x_{pulse}(t_n-mT_1)\ . \qquad (27.5)$$

There are N points t_n per period T_1. If we rewrite (27.2) evaluated at these points, taking ω_1 from (27.3) and setting $t = t_n = n\Delta t$, we get

$$x(t_n) \ = \ \sum_{m=-\infty}^{\infty} c_m \, e^{+imn(2\pi/N)} \ . \qquad (27.6)$$

So far we haven't really done anything except examine x(t) at some sample points.

Following an approach similar to that of Section 22, consider now the following *non*-equation,

$$c_m \ \neq \ (1/T_1) \sum_{n=-\infty}^{\infty} \Delta t \, x_{pulse}(t_n) \, e^{-im\omega_1 t_n} \ . \qquad (27.7)$$

Only in the limit $\Delta t \rightarrow 0$ does (27.7) become an equality, since it then reproduces (27.1). So let's define something *new* called c'_m that *is* equal to the right side of (27.7) for a specific finite $\Delta t = T_1/N$:

$$c'_m \ \equiv (1/T_1) \sum_{n=-\infty}^{\infty} \Delta t \, x_{pulse}(t_n) \, e^{-im\omega_1 t_n} \ . \qquad (27.8)$$

Using (27.3) this becomes

$$c'_m \ \equiv \ (1/N) \sum_{n=-\infty}^{\infty} x_{pulse}(t_n) \, e^{-imn(2\pi/N)} \ . \qquad \text{projection = transform} \qquad (27.9)$$

Recall that $x_{pulse}(t)$ is usually taken to be a pulse which vanishes outside a range of width T_1, and in this case the sum in (27.9) has only N non-vanishing terms.

In the Fourier Series world, one can have any number of unique c_m coefficients. For the c'_m in (27.9) this is no longer true. There are in fact only N unique values of c'_m because they keep repeating. This is because (27.9) implies that

$$c'_{m+kN} \ = c'_m \qquad \text{for any integer k} \qquad (27.10)$$

due to the fact that $e^{-i(kN)n(2\pi/N)} = e^{-ikn(2\pi)} = 1$.

So c'_m is a periodic digital sequence of period N.

We claim now (to be proven in section (b) below) that the correct expansion of pulse train x(t) to accompany projection (27.9) is the following:

$$x(t_n) \ = \ \sum_{m=0}^{N-1} c'_m \, e^{+imn(2\pi/N)} \ . \qquad \text{expansion = inverse transform} \qquad (27.11)$$

Due to the periodicity of c'_m shown in (27.10), the expansion (27.11) can also be written as

$$x(t_n) = \begin{cases} \sum_{m=-N/2}^{N/2-1} c'_m e^{+imn(2\pi/N)} & \text{N even} \\ \sum_{m=-(N-1)/2}^{(N-1)/2} c'_m e^{+imn(2\pi/N)} & \text{N odd} \end{cases} \qquad (27.12)$$

<u>Proof of (27.12):</u> For N even write the claimed result separating off the negative part of the series,

$$\sum_{m=-N/2}^{N/2-1} c'_m e^{+imn(2\pi/N)} = \sum_{m=-N/2}^{-1} c'_m e^{+imn(2\pi/N)} + \sum_{m=0}^{N/2-1} c'_m e^{+imn(2\pi/N)} .$$

In the first term replace m by m' = m+N to get

$$\sum_{m=-N/2}^{-1} c'_m e^{+imn(2\pi/N)} = \sum_{m'=N/2}^{N-1} c'_{m'-N} e^{+i(m'-N)n(2\pi/N)} = \sum_{m'=N/2}^{N-1} c'_{m'} e^{+i(m')n(2\pi/N)}$$

where we have used (27.10) to say $c'_{m'-N} = c'_{m'}$ and $e^{+i(-N)n(2\pi/N)} = 1$. Changing m'→m we then write the our two-term sum as

$$\sum_{m=-N/2}^{N/2-1} c'_m e^{+imn(2\pi/N)} = \sum_{m=N/2}^{N-1} c'_m e^{+imn(2\pi/N)} + \sum_{m=0}^{N/2-1} c'_m e^{+imn(2\pi/N)} = \sum_{m=0}^{N-1} c'_m e^{+imn(2\pi/N)} .$$

But this the sum in (27.11), so we have proven (27.12) for N even. The proof for odd N is similar and it is left to the reader.

We can now verify that our new transform approaches the Fourier Series Transform (27.1) and (27.2) in the limit N→∞. The first line below is the large N (small Δt) limit of (27.9), while the second line is the limit of (27.12) for even or odd N where N>>1 (use is made of the relations in (27.3) and $\lim_{N\to\infty} t_n = \lim_{N\to\infty} (n\Delta t) = t$) :

$$\lim_{N\to\infty} c'_m = \lim_{N\to\infty} \{(1/T_1) \sum_{n=-\infty}^{\infty} \Delta t\, x_{pulse}(t_n)\, e^{-im\omega_1 t_n}\} = (1/T_1) \int_{-\infty}^{\infty} x_{pulse}(t)\, e^{-im\omega_1 t} = c_m$$

$$\lim_{N\to\infty} x(t_n) = \lim_{N\to\infty} \{ \sum_{m=-N/2}^{N/2} c'_m e^{+im\omega_1 t_n}\} = \sum_{m=-\infty}^{\infty} c_m e^{+im\omega_1 t} = x(t) \quad ,$$

This new transform pair (27.9) and (27.11) we shall call the **Discrete Fourier Transform of a Pulse Train**, as opposed to the Fourier Series Transform pair given in (27.1) and (27.2). We reserve the term Discrete Fourier Transform to refer to the transform of an isolated pulse. As we shall see in section (c) below, for this normal DFT, the sum in (27.9) becomes finite.

In the Discrete Fourier Transform, time is "discrete", only t_n appear, and therefore we only see functions evaluated at these discrete time points,

$$x_n \equiv x(t_n)$$
$$x_{pulse,n} \equiv x_{pulse}(t_n)$$

In contrast, the Fourier Series Transform has a continuous time variable t and functions $x_{pulse}(t)$ and pulse train x(t) appear.

Both transforms have discrete *spectra* as indicated by c_m and c'_m and this is because in both cases the pulse train is a periodic function.

Just as a reminder, with the Digital Fourier Transform we dealt with sample sequences like $y_n = y(t_n) = y(nT_1)$ where T_1 was the spacing between pulses composing a pulse train. In such a sequence, there is only one sample per T_1 period. In contrast, with our current Discrete Fourier Transform discussion, $y_n = y(t_n) = y(nT_1/N)$ and there are N samples per T_1 time period. In both cases one could argue that the sequence is a set of digital or discrete values, so the transform names are somewhat arbitrary.

(b) Proof of the Discrete Fourier Transform for a Simple Pulse Train x(t)

To show this transform really works, we insert (27.9) for c'_m into (27.11),

$$x(t_n) = \sum_{m=0}^{N-1} c'_m e^{+imn(2\pi/N)} = \sum_{m=0}^{N-1} [(1/N) \sum_{k=-\infty}^{\infty} x_{pulse}(t_k)\, e^{-imk(2\pi/N)}]\, e^{+imn(2\pi/N)}$$

$$= (1/N) \sum_{k=-\infty}^{\infty} x_{pulse}(t_k) \sum_{m=0}^{N-1} e^{+im(2\pi/N)(n-k)} \quad . \qquad (27.13)$$

We now quote an obscure identity proven in Appendix B which says

$$\sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = N \sum_{m=-\infty}^{\infty} \delta_{s,mN} \qquad N > 0 \qquad s = \text{integer} \quad . \qquad \textit{(B.1)}$$

This is a discrete version (s = integer) of (13.2) (s = real) which we quote for comparison

$$\sum_{m=-\infty}^{\infty} e^{ims} = \sum_{m=-\infty}^{\infty} 2\pi\delta(s - 2\pi m) \qquad -\infty < s < \infty \quad . \qquad \textit{(13.2)}$$

Setting s = n-k, (B.1) says [since $\delta_{n-k,mN} = \delta_{k,n-mN}$]

$$\sum_{m=0}^{N-1} e^{+im(2\pi/N)(n-k)} = N \sum_{m=-\infty}^{\infty} \delta_{k,n-mN} \quad . \qquad (27.14)$$

Then we find that

$$x(t_n) = (1/N) \sum_{k=-\infty}^{\infty} x_{pulse}(t_k)\ N \sum_{m=-\infty}^{\infty} \delta_{k,n-mN} = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_{pulse}(t_k)\, \delta_{k,n-mN}$$

$$= \sum_{m=-\infty}^{\infty} x_{pulse}(t_{n-mN}) = \sum_{m=-\infty}^{\infty} x_{pulse}(t_n - mT_1) \ . \qquad (27.15)$$

Since this reproduces the pulse train (27.5), we conclude that indeed (27.11) is the expansion that accompanies the projection (27.9).

At this point, we make a box to summarize the Discrete Fourier Transform of a simple pulse train:

Discrete Fourier Transform of a Simple Pulse Train (27.16)

1. Let $x_{pulse}(t)$ be any reasonable pulse. Construct a simple pulse train x(t) with spacing T_1:

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) \qquad \Rightarrow \qquad x(t + mT_1) = x(t) \qquad \textit{(27.5)}$$

By its construction, x(t) is periodic with period T_1. If x(t) is a *known* periodic function of period T_1, a candidate for $x_{pulse}(t)$ is x(t) over any one period (and zero elsewhere).

2. Break up each T_1 interval into N steps of width $\Delta t = T_1/N$. Let $t_n = n\Delta t = (n/N)T_1$.

3. Define the Discrete Fourier Transform coefficients c'_m by this projection = transform:

$$c'_m \equiv (1/N) \sum_{n=-\infty}^{\infty} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \qquad m = \text{integer} \qquad \textit{(27.9)}$$

Only N of these are unique because c'_m is periodic in index m with period N:

$$c'_{[m+nN]} = c'_m \qquad n = \text{any integer} \qquad \textit{(27.10)}$$

4. The pulse train at sample points t_n is then given by this expansion = inversion:

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \qquad \textit{(27.11) but see also (27.12)}$$

From this last result we may confirm that $x(t_n + mT_1) = x(t_n)$ so x is indeed periodic.

(c) The Discrete Fourier Transform for an Arbitrary Pulse

The results of box (27.16) apply to any sampled pulse $x_{pulse}(t)$. We could consider, for example, the set of $x_{pulse}(t_n)$ functions which are non-zero only for $t_n = n\Delta t$ lying inside some limited temporal range indicated by $A \le n \le B$. For such functions (27.8) will have the form,

$$c'_m \equiv (1/N) \sum_{n=A}^{B} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \quad . \qquad (27.17)$$

The inversion formula continues to be (27.11)

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \quad . \qquad \textit{(27.11)}$$

If the range $(A\text{-}B)\Delta t > T_1$, the above stated transform is valid, but the pulse $x_{pulse}(t_n)$ cannot in this case have an arbitrary shape. This is because (27.11) forces $x(t_n + T_1) = x(t_n)$, meaning $x(t)$ is periodic. For example, if $(A\text{-}B)\Delta t \approx 1.3\, T_1$, then the portion of $x_{pulse}(t_n)$ in $(T_1, 1.3T_1)$ must be a replication of the portion of $x_{pulse}(t_n)$ in $(0, 0.3T_1)$. If we want a DFT pulse transform that allows for arbitrary pulse shape, we must restrict A and B so that $(A\text{-}B)\Delta t \le T_1$. We can of course consider pulses which are restricted to $(A\text{-}B)\Delta t < T_1$ to be special cases of pulses defined on $(A\text{-}B)\Delta t = T_1$, where we just add zero padding to arrive at the interval T_1.

Therefore we restrict A,B so that $(A\text{-}B)\Delta t = T_1$ or $(A\text{-}B) = T_1/\Delta t = N$.

In this way, we arrive at this special case of the transform of the box (27.16) which applies to an arbitrary pulse of width T_1 (that is, a pulse having only N discrete values)

$$c'_m \equiv (1/N) \sum_{n=0}^{N-1} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \qquad m = 0,1...N\text{-}1 \qquad (27.18)$$

$$// \ (2\pi/N) = \omega_1\Delta t$$

$$x_{pulse}(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \qquad n = 0,1,...N\text{-}1 \qquad \textit{(27.11)} \qquad (27.19)$$

This is the official **Discrete Fourier Transform (DFT)**. Due to the periodicity property (27.10), the sum in (27.17) could be taken over any set of N adjacent steps, and without loss of generality we take these N steps to be 0,1,...N-1. If one were to regard the pulse as being translated to some other set of N steps like n = -3,-2,-1,0,1,... N-4, the coefficients c'_m would be exactly the same apart from a simple m-dependent phase. For example, let x'_{pulse} be the translated pulse. Then,

$$d'_m \equiv (1/N) \sum_{n=-3}^{N-4} x'_{pulse}(t_n)\, e^{-imn(2\pi/N)} = (1/N) \sum_{n=-3}^{N-4} x_{pulse}(t_{n-3})\, e^{-imn(2\pi/N)}$$

$$= (1/N) \sum_{n'=0}^{N-1} x_{pulse}(t_{n'})\, e^{-im(n'-3)(2\pi/N)} \qquad // \; n' = n+3$$

$$= e^{+i3m(2\pi/N)} \left\{ (1/N) \sum_{n'=0}^{N-1} x_{pulse}(t_{n'})\, e^{-imn(2\pi/N)} \right\} = e^{+i3m(2\pi/N)}\, c'_m$$

$$= e^{+i3(m\omega_1)\Delta t}\, c'_m \qquad (27.20)$$

This result is a reflection in the current context of the time-shift rule (12.1) which we restate here as

$$x(t + 3\Delta t) \quad \leftrightarrow \quad e^{+i3\omega\Delta t}\, X(\omega) \qquad \textit{(12.1)}$$

Note that c'_m refers to the spectral frequency $m\omega_1$.

We now summarize the DFT in a box:

Discrete Fourier Transform for an Arbitrary Pulse (27.21)

1. Let $x_{pulse}(t)$ be an arbitrary reasonable pulse defined for t in $(0,T_1)$.

2. Break up T_1 into N steps of width $\Delta t = T_1/N$. These relationships hold

$$\Delta t = T_1/N \qquad \omega_1 \equiv (2\pi/T_1) = 2\pi/(N\Delta t)$$
$$T_1 = N\,\Delta t \qquad (2\pi/N) = \omega_1\Delta t$$
$$t_n = n\Delta t \qquad \omega_1 t_n = n\,\omega_1\Delta t = n\,(2\pi/N) \qquad \textit{(27.3)}$$

Thus, the sequence values of interest are $x_{pulse}(t_n)$ for n = 0,1,2...N-1.

3. Define the Discrete Fourier coefficients c'_m by this projection = transform:

$$c'_m \equiv (1/N) \sum_{n=0}^{N-1} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} \qquad m = 0,1...N-1 \qquad \textit{(27.18)}$$

4. The accompanying expansion = inverse transform is given by

$$x_{pulse}(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \qquad n = 0,1,...N-1 \qquad \textit{(27.19)}$$

In the above box one could of course replace $x_{pulse}(t_n)$ by some generic function $x(t_n)$ defined on $(0,T_1)$, and one could go further and write $x(t_n)$ as x_n to get this more common statement of the DFT:

$$c'_m \equiv (A/N) \sum_{n=0}^{N-1} x_n e^{-imn(2\pi/N)} \qquad m = 0,1...N-1 \qquad \text{projection = transform}$$

$$x_n = (1/A) \sum_{m=0}^{N-1} c'_m e^{+imn(2\pi/N)} \qquad n = 0,1,...N-1 \qquad \text{expansion = inverse transform} \qquad (27.22)$$

Here we have added an arbitrary constant A to the first equation and 1/A to the second which maintains the validity of the DFT transform pair. If one takes A=N, the 1/N factor moves to the second equation. Another choice is $A = \sqrt{N}$ to make the two equations symmetrical. The transform pair is also valid if the phase signs are switched, just as with the Fourier Integral Transform. This would be associated with a version of (B.1) having the opposite phase sign obtained by just complex conjugating (B.1).

(d) Comments on the Discrete Fourier Transform

One might wonder about the purpose of the Discrete Fourier Transform of a Pulse Train, and its relation to earlier transforms. This can be illuminated by a simple set of pictures.

First, go back to the Section 2 analysis of a making a pulse train x(t) by superposing shifted copies of pulse $x_{pulse}(t)$. Imagine, as in the discussion at the end of Section 14, that $x_{pulse}(t)$ is a Gaussian which of necessity extends beyond the domain of one period T_1. Here is a picture of this pulse:

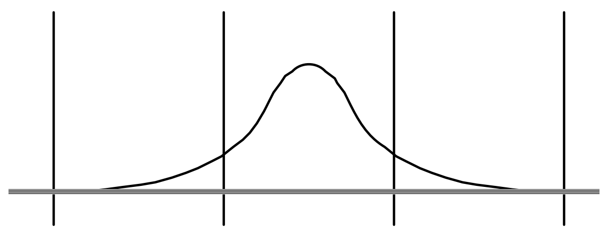


Figure 27.1. The pulse $x_{pulse}(t)$. Bars are distance T_1 apart. Fig 27.1

If we now superpose these pulses, we get the following pulse train x(t), as was shown in Fig 14.2,

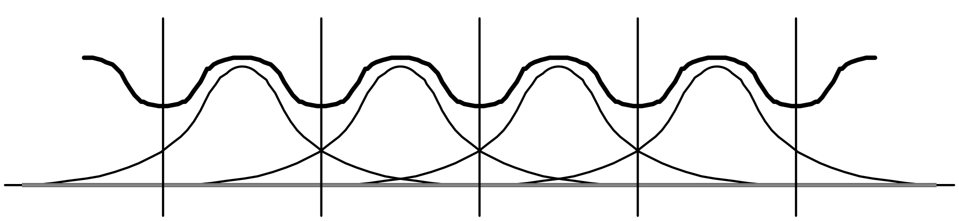


Figure 27.2. The function x(t) is the heavy curve. It is the sum of the gaussians. Fig 27.2

This picture gives us a chance to repeat a point made earlier, namely that $x_{pulse}(t)$ is not unique. One could use instead a portion of the heavy curve between any adjacent pair of bars.

The heavy curve is our pulse train, and we have chosen the most complicated case, that where the pulses overlap. Typically they do not overlap.

Now, the heavy curve is a periodic continuous function of time x(t), and it has in principle an infinite set of Fourier Series coefficients c_m. These coefficients are really determined from the underlying pulse $x_{pulse}(t)$. In general, it takes an infinite number of time points to represent the smooth function $x_{pulse}(t)$, so there are an infinite number of coefficients c_m in the "transformed space" where these coefficients live.

We know that the transform of the Gaussian $x_{pulse}(t)$ is a Gaussian $X_{pulse}(\omega)$, and we know that the Fourier coefficients are given by

$$c_m = (1/T_1)\, X_{pulse}(m\omega_1) \qquad \text{where } \omega_1 = 2\pi/T_1 .$$

One can visualize (see Fig 14.1) the infinite set of the c_m as tracing the envelope of this gaussian $X_{pulse}(\omega)$. Of course it may happen that many of the c_m vanish if $x_{pulse}(t)$ has simple harmonic content. The point is that there *could* be an infinite number of c_m.

This infinitude matches the infinitude of real points along the pulse $x_{pulse}(t)$. If we consider the regular Fourier integral spectrum $X_{pulse}(\omega)$ in its own right, we again have an infinitude of complex numbers needed to describe the pulse in the transformed space, subject to $X(-\omega) = [\,X(\omega)]^*$ of (7.3) which knocks down this complex double infinity to a single infinity, balancing the time side of the transform.

Having said all this, we are now ready to move from analog to digital. Consider the same pulse $x_{pulse}(t)$ evaluated only at the discrete points t_n, so the pulse is now represented by this sequence of numbers:

$$x_{pulse}(t_n) \qquad\qquad t_n = n\,\Delta t$$

and we assume that there are N sample points in each period T_1. In our figures below, N = 8, and we approximate the tail of the Gaussian with a few extra points.

Here then is a picture of the set of numbers $x_{pulse}(t_n)$ which describe our pulse:

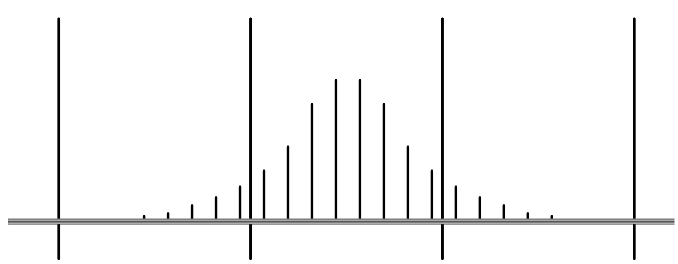


Figure 27.3. Pulse is now a set of 18 numbers x_n(pulse) . N = 8 Fig 27.3

Now as before, build a digital pulse train by superposing pulses:

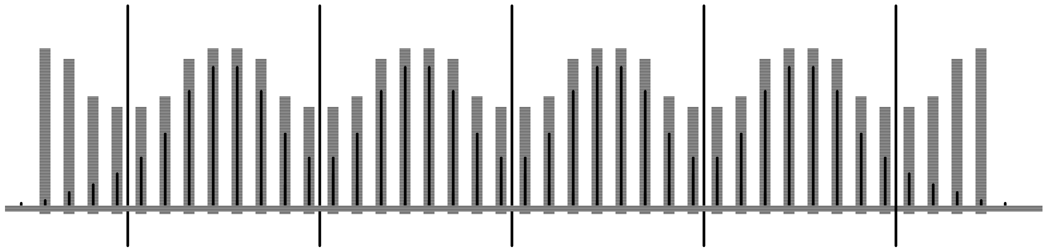


Figure 27.4. Digital pulse train represented by the fat hatched bars. Fig 27.4

In this figure, the thin dark bars are the numbers which describe the pulse. The fat hatched bars represent the sum of the thin bars -- remember that we have overlap here.

Note that the resulting sequence -- the fat bars -- form a periodic sequence, just as we had a periodic function given by the heavy curve in Figure 27.2. Note also that again we could have used an "equivalent pulse sequence" here consisting of just the set of 8 fat bars in one interval.

Thus, although our original pulse contained 18 numbers, the minimal pulse contains only 8 numbers. If we now compute the Discrete Fourier Transform coefficients c_m' according to the formula in box (26.15),

$$c'_m \equiv (1/N) \sum_{n=-\infty}^{\infty} x_{pulse}(t_n)\, e^{-imn(2\pi/N)}$$

we find that only 8 of the c'_m are unique because of the translation rule shown in the same box. Select those with m = 0,1,2,3,4,5,6,7. We should be happy to find that it takes only 8 numbers in the transform space (where the c_m' live) to represent the 8 numbers in the time domain which represented our pulse in its minimal representation -- the 8 fat bars in period T_1. For this minimal pulse, there are only 8 non-vanishing terms in the above sum. Thus, the c_m' are related to the eight fat bar heights by a set of numbers which form an 8x8 symmetric matrix, namely

$$M_{mn} = (1/N)\, e^{-imn(2\pi/N)}$$

in terms of which we have

$$c'_m = \Sigma_m\, M_{mn}\, x_{pulse}(t_n)$$

or

$$\mathbf{c}' = M\, \mathbf{x}_{pulse} \qquad (27.18)$$

$$\mathbf{x}_{pulse} = M^{-1}\, \mathbf{c}' = N\, M^*\, \mathbf{c}' \qquad (27.19)$$

Of the 64 matrix elements of M, only 8 are unique, and these 8 elements lie equally spaced on a circle of radius (1/N) in the complex plane.

As a possible application of the Discrete Fourier Transform (DFT), consider some sort of digital circuit that puts out a set of numbers that repeat after every N numbers. Perhaps this is what a scrambler does with a constant input. In this case, one can think of the set of N numbers as tracing the envelope of a pulse

$x_{pulse}(t)$. The appropriate "frequency domain" transform of this repeating sequence of numbers is the DFT. In the frequency domain we get a finite set of N numbers c_m' as the transform.

Just as with regular Fourier series coefficients, the DFT coefficients c'_m are a measure of the frequency content of the signal. Recall that the pulse train x(t) is mapped out by:

$$x(t_n) = \sum_{m=0}^{N-1} c'_m e^{+imn(2\pi/N)} = \sum_{m=0}^{N-1} c'_m e^{+im\omega_1 t_n} \qquad \omega_1 = (2\pi/T_1)$$

One should think of n taking lots of values and tracing out the envelope of the function x(t). Clearly, coefficient c'_m is the weight of frequency component $m\omega_1$. So c'_0 measures the DC component, and c'_1 measures the amount of frequency component ω_1 and so on.

There is a limit on how high a frequency component one can have. Consider a sine wave with period Δt = the sample spacing. It would have the same value at every sample point, and would thus show up in the DC component. The frequency corresponding to period Δt is $\omega_N = N\omega_1$. This is why $c'_N = c'_0$. In a similar fashion, potential frequencies ω_n with $n>N$ are also "aliased" down into lower frequencies according to the translation rule for the c'_m. Thus, the highest frequency we can really have is $(N-1)\omega_1$.

So this gives a reasonable "Fourier explanation" of why there are a finite number of distinct c'_m coefficients involved in the spectral expansion above for $x(t_n)$.

We repeat one more time an important fact stressed earlier: as N (the number of sample points per T_1 interval) increases, the number of DFT coefficients c'_m increases as well, and these c'_m becomes closer and closer to the Fourier Series coefficients c_m. In the limit $N \rightarrow \infty$, $c'_m = c_m$, and the DFT and the Fourier Series exactly align.

For finite N, the c'_m differ from the c_m in exactly the same way that the area under a stepwise approximated curve differs from the area under the smooth curve. This fact follows directly from the definitions of the c'_m and c_m.

Chapter 4: Some Practical Topics

This chapter applies the results of earlier chapters to a few simple test situations and applications. By providing some wordy discussion of seemingly mundane topics, we attempt to prop up our so-far mostly mathematical approach to Fourier analysis. It is in matters like these that one's understanding is really put to the test.

28. Do FIR filters have linear phase?

We shall show in two different ways that a FIR filter has linear phase provided it has symmetric coefficients. The first method is direct, the second more intuitive.

Method 1

We saw in (21.5) how a digital filter is represented by a set of numbers b_n. Recall the Z transform projection,

$$B''(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n} \quad . \qquad \text{// FIR filter}$$

Since z lies on the unit circle, we may represent it as $z = e^{i\theta}$ as in (24.1). Thus,

$$B''(e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{-in\theta} \quad . \qquad \text{// FIR filter}$$

Assume that b_n is a finite set $b_0, b_1, b_2....b_N$. Then,

$$B''(z) = \sum_{n=0}^{N-1} b_n z^{-n} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... b_{N-1}z^{-(N-1)} \quad . \qquad (28.1)$$

Assume next that the set of b_n is "symmetric" such that $b_0 = b_{N-1}$, $b_1 = b_{N-2}$ and so on.

It is not hard to obtain our conclusion using general N, but it is a lot easier to see what is going on if we pick some sample N values.

Let N = 5. Then we have

$$\begin{aligned} B''(z) = \sum_{n=0}^{4} b_n z^{-n} &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3z^{-3} + b_4z^{-4} \\ &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_1z^{-3} + b_0z^{-4} \qquad \text{// assume symmetric} \\ &= z^{-2} (b_0z^2 + b_1 z + b_2 + b_1z^{-1} + b_0z^{-2}) \\ &= z^{-2} [b_0(z^2 + z^{-2}) + b_1(z + z^{-1}) + b_2] \quad . \end{aligned}$$

Now set $z = e^{i\theta}$ and continue along,

$$= e^{-2i\theta}\,[b_0(e^{2i\theta} + e^{-2i\theta}) + b_1(e^{i\theta} + e^{-i\theta}) + b_2]$$
$$= 2e^{-2i\theta}\,[b_0\cos(2\theta) + b_1\cos(\theta) + b_2]\,. \qquad 2 = (N-1)/2$$

The phase of this filter is -2θ. If we used N = 7, a repeat of the above analysis would give

$$B''(z) = 2e^{-3i\theta}\,[b_0\cos(3\theta) + b_1\cos(2\theta) + b_2\cos(\theta) + b_3] \qquad 3 = (N-1)/2$$

with a phase of -3θ. For a general odd value of N, the z phase comes out being -i[(N-1)/2]θ .

Now consider even values of N. For N = 4 we have

$$B''(z) = \sum_{n=0}^{3} b_n z^{-n} = b_0 + b_1 z^{-1} + b_1 z^{-2} + b_0 z^{-3}$$
$$= z^{-1.5}\,(b_0 z^{1.5} + b_1 z^{.5} + b_1 z^{-.5} + b_0 z^{-1.5})$$
$$= z^{-1.5}\,[b_0(z^{1.5} + z^{-1.5}) + b_1 (z^{.5} + z^{-.5})]$$
$$= 2e^{-i1.5\theta}[b_0\cos(1.5\theta) + b_1\cos(0.5\theta)]\;. \qquad 1.5 = (N-1)/2$$

For N = 6 the result would be

$$= 2e^{-i2.5\theta}[b_0\cos(2.5\theta) + b_1\cos(1.5\theta) + b_2 \cos(0.5\theta)] \qquad 2.5 = (N-1)/2\;.$$

For a general even value of N, the phase comes out being -[(N-1)/2]θ which is the same as the phase for the general odd N value. Thus we have shown that, for general N, and using θ = ωΔt from (24.1),

$$B''(z) = 2\, e^{-i\omega\Delta t(N-1)/2}\ [\text{ real sum of cosine terms }] \qquad (28.2)$$

According to the definition of "filter phase" in (21.14), our filter B"(z) has

$$\text{phase} = +\,[(N-1)/2]\,\Delta t\,\omega\,. \qquad (28.3)$$

Since this phase is linear in ω, our symmetric-coefficient digital FIR filter has "linear phase". Using the same definition of group delay used for an analog filter in (21.19), the group delay for such a linear phase digital filter is

$$\tau_d = d(\text{phase})/d\omega = \Delta t(N-1)/2 = \text{a constant} \qquad (28.4)$$

so we expect a symmetric FIR filter to exhibit good fidelity when it acts on an input signal.

Method 2

Consider the Fourier integral spectrum X(ω) of a real-valued pulse x(t) that is symmetrical and centered at t=0. Since x(-t) = x(t), we can fold the negative portion of the dt integration in (1.1) over to the positive side. Doing this gives x(t) times $e^{-i\omega t} + e^{+i\omega t}$ = 2cos(ωt). Thus, everything is *real*, and X(ω) must therefore be real. We have already seen several examples of this: δ(t) gives X(ω) = 1, a square pulse gives (Aτ) sinc(ωτ/2).

If we displace the pulse to the right by some amount of time MΔt, then X(ω) is no longer real, it picks up the usual shift phase from (12.1) which here would be exp(-iωMΔt).

We now construct a digital filter O"(z) = B"(z)I"(z). The input will be a unit pulse at time 0 so $i_n(t)$ = $\delta_{n,0}$ and therefore I"(z) = 1 from (24.8) and (24.9) with m=0. The output of the filter is then O"(z) = B"(z). Consider an N=3 filter with B"(z) = a + bz^{-1} + az^{-2}. This corresponds to output signal o(0) = a, o(Δt) = b and o(2Δt) = a. Since this is a symmetric pulse centered at t = Δt, we know from the previous paragraph that the spectrum of this pulse has a phase $e^{-i\Delta t}$ relative to the real spectrum of a similar pulse centered at t = 0. Next, consider an N=4 filter with coefficients a,b,b,a. The output will be sequence a,b,b,a centered at t = (3/2)Δt, so its spectral phase will be $e^{-i(3/2)\Delta t}$ relative to that of similar pulse centered at t = 0. In both cases, we see that the output pulse is centered at t = [(N-1)/2] Δt, so this must be the group delay of a symmetric filter with N coefficients. The filter phase must then be the function φ = [(N-1)/2] Δt ω, which is linear in ω, in agreement with the result of the more formal Method 1.

29. A Simple Digital Low-Pass Filter

This section describes a particular implementation of a digital low-pass filter. There is a whole world of such filters, and this design is meant only as an illustration. In Section 30 the filter described here will be used as a 4x oversampling interpolation filter for a D/A converter output design.

The ideal "brick wall" filter has this spectrum,

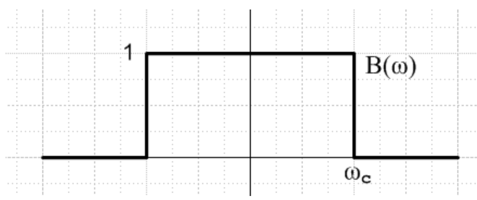


Fig 29.1

Recall our box-shaped pulse in the time domain (height 1, width τ) and its spectrum

$$x(t) = [\ \theta(t + \tau/2) - \theta(t - \tau/2)\ ] \qquad (9.1)$$

$$X(\omega) = \tau\ \mathrm{sinc}(\omega\tau/2)\ . \qquad (9.2)$$

For this x(t), (1.1) gives the X(ω) shown. If we try $X(\omega) = [\ \theta(\omega + \omega_c) - \theta(\omega - \omega_c)\ ]$ in (1.2), we know the result will be $(1/2\pi) * 2\omega_c\ \mathrm{sinc}(t\omega_c)$, just swapping the variables $t \leftrightarrow \omega$ and $\tau/2 \rightarrow \omega_c$. We thus obtain the following brick wall filter B(ω) of Fig 29.1 and it associated time-domain pulse shape b(t),

$$B(\omega) = [\ \theta(\omega + \omega_c) - \theta(\omega - \omega_c)\ ] \qquad (29.1)$$

$$b(t) = (\pi/\omega_c)\ \mathrm{sinc}(\omega_c t)\ . \qquad \text{// b(t) is even in t} \qquad (29.2)$$

As mentioned in Comment (1) at the end of Section 3, since b(t) is a filter kernel, it has dimensions of inverse time and the filter spectrum (transfer function) B(ω) is dimensionless.

Recall the Convolution theorem (3.6),

$$o(t) = \int_{-\infty}^{\infty} dt'\ b(t-t')i(t') \qquad \Leftrightarrow \qquad O(\omega) = B(\omega)\ I(\omega) \qquad (3.6)$$

where i(t) is the input to a filter and o(t) the output. Using i(t) = δ(t), we get o(t) = b(t), so b(t) is the impulse response of the filter. [In this special situation, we are not following our Section 3 Comment (1) convention since this i(t) has dimensions of inverse time instead of being dimensionless.]

A digital filter approximating (3.6) has this form,

$$o(t_n) = \sum_{m=-\infty}^{\infty} \Delta t\ \ b(t_n - t_m)\ i(t_m) \qquad \text{where } t_n = n\,\Delta t\ . \qquad (21.4)$$

or

$$o_n = \sum_{m=-\infty}^{\infty} \Delta t\, b_{n-m}\, i_m \quad . \tag{29.3}$$

If we set the input to a digital unit impulse at t = 0, $i(t_m) = i_m = \delta_{m,0}$, then the output is

$$o_n = \Delta t\, b_n \tag{29.4}$$

so $\Delta t\, b_n$ is the unit impulse response of the digital filter. Recalling from (3.2) the symmetry of the convolution equation (or just set $m' \equiv n\text{-}m$) we can write (29.3) instead as

$$o_n = \sum_{m=-\infty}^{\infty} \Delta t\; i_{n-m}\; b_m \,. \tag{29.5}$$

In equations (29.3) and (29.5) we think of i_n and o_n as being dimensionless, and b_n as having dimensions of inverse time so $\Delta t\, b_n$ is dimensionless. In (24.6) we used $h_n \equiv \Delta t\, b_n$ but here we stick with b_n.

Example: We assume these parameters, since the resulting filter will be useful later on :

$T_1 = 1$
$\omega_1 = 2\pi/T_1 = 2\pi$
$\omega_c = \omega_1/2 = \pi$ // Nyquist rate, see end of Section 20
$\Delta t = T_1/4 = 1/4$ // filter will be clocked at 4x rate (29.6)

Maple computes the b_m from (29.2) as follows (tiny offset added to avoid divide by zero in the hand-made sinc function)

```
T1 := 1; w1 := 2*Pi/T1; wc := w1/2:  dt := T1/4:
sinc := x -> sin(x)/(x):
for i from -10 to 10 do b[i] := evalf((Pi/wc)*sinc(wc*i*dt+.0000001)) od:
for i from 0 to 10 do printf("b[%d] = b[%d] = %6.3f\n", i, -i,b[i]) od;
```

```
b[0]  = b[0]  =  1.000
b[1]  = b[-1]  =   .900
b[2]  = b[-2]  =   .637
b[3]  = b[-3]  =   .300
b[4]  = b[-4]  =  -.000
b[5]  = b[-5]  =  -.180
b[6]  = b[-6]  =  -.212
b[7]  = b[-7]  =  -.129
b[8]  = b[-8]  =   .000
b[9]  = b[-9]  =   .100
b[10]  = b[-10]  =   .127
```
(29.7)

The filter has symmetric coefficients $b_{-n} = b_n$ since b(t) in (29.2) is even in t, so it will exhibit a linear phase response as discussed in Section 28. This in turn means a constant group delay as in (28.4).

We can verify the locations of these points on the sinc curve,

```
p1 := plot((Pi/wc)*sinc(wc*n*dt+.001),n=-10..10):
s := seq([k,b[k]],k=-10..10):
p2 := pointplot({s},symbol = circle ):
display([p1,p2], xtickmarks = 20);
```

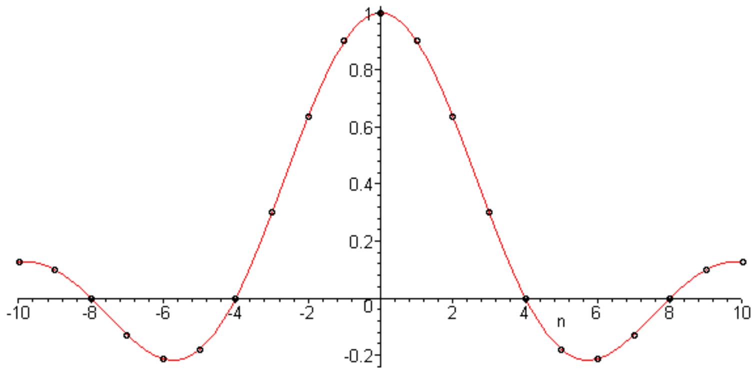


Fig 29.2

The digital filter implementation using (29.5) is just this equation ($\Delta t = 1/4$),

$$4o_n = \sum_{m=-10}^{10} i_{n-m}\, b_m = i_n b_0 + \sum_{m=1}^{10} i_{n-m}\, b_m + \sum_{m=-1}^{-10} i_{n-m}\, b_m$$

$$= i_n b_0 + \sum_{m=1}^{10} i_{n-m}\, b_m + \sum_{m=1}^{10} i_{n+m}\, b_m \qquad\qquad // \text{ since } b_{-m} = b_m$$

$$= i_n b_0 + \sum_{m=1}^{10} [i_{n-m} + i_{n+m}]\, b_m$$

$$= i_n b_0 + [i_{n-1} + i_{n+1}]\, b_1 + [i_{n-2} + i_{n+2}]\, b_2 + ... + [i_{n-10} + i_{n+10}]\, b_1 \quad . \qquad (29.8)$$

This equation is implemented in the following piece of hardware,

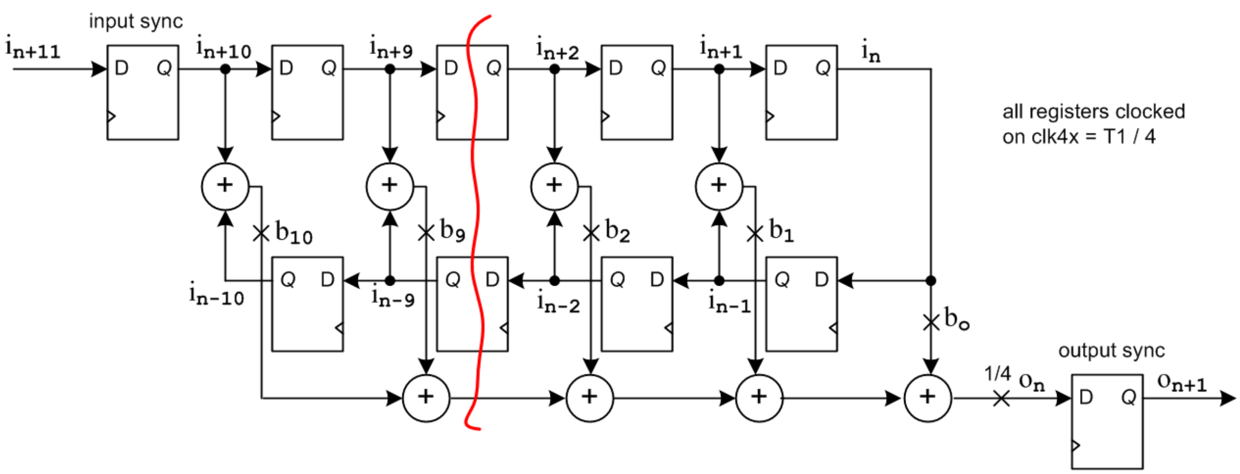


Fig 29.3

Notice that the registers on the top march samples left to right, while those on the bottom go right to left. The clock lines are not drawn; all registers are clocked with period $\Delta t = T_1/4 = 1/4$. The registers (D flip-flops) sometimes appear as boxes containing z^{-1} as in Fig 24.4. If a register input is i_{n+1} in the middle of a clock period, that register's output is the previously clocked sample i_n. Usually the clock is a square wave signal and the registers transfer input to output on the positive clock edges of the square wave. The circled plus signs are adders, while lines marked with an X indicate multiplication by the constant appearing next to the X. All lines indicate busses containing some number of bits used to represent the digital signals, perhaps 8, 10 or 12.

An actual design might be done a bit differently using pipelining registers to avoid the large combinatoric delay built up through the long string of adders at the bottom (if speed is an issue).

We now wish to compute the spectrum ("transfer function") of this digital filter to see how close it comes to being a "brick wall" with cutoff at ω_c. Basically, we want to compute the Digital Fourier Transform spectrum associated with the finite sequence of samples b_m. Recall that $\Delta t\, b_m$ is the dimensionless impulse response of the filter.

$$b_m = (\pi/\omega_c)\mathrm{sinc}(\omega_c m\Delta t) \qquad m = -10 \text{ to } 10, \quad \text{else } 0 \qquad 21 \text{ "taps"} \qquad (29.9)$$

We know that if we include terms from $m=-\infty$ to $m=+\infty$, we shall obtain for the Digital Fourier Transform spectrum $B'(\omega)$ an exact brick wall box-shaped filter with image boxes going off to the left and right. But for m limited to the range (-10,10), which makes use of 21 b_m coefficients (a 21 "tap" filter), we expect to get only an approximation to Fig 29.1. From box (23.5),

$$B'(\omega) \equiv (T_1/4) \sum_{n=-10}^{10} b_n\, e^{-i\omega n T_1/4} \ . \qquad (29.3)$$

Here is a plot of the central peak for a filter of 21 taps (blue) compared with one of 101 taps (red). Notice that the cutoff frequency is at $\omega_c = \pi$, and as usual plots are of $|B'(\omega)|$, so the ringing on both sides of the peak is "rectified",

```
for i from -50 to 50 do b[i] := evalf((Pi/wc)*sinc(wc*i*dt+.0000001)) od:
asum := w -> (1/4)*sum(b[n]*exp(-I*n*w*T1/4),n=-50..50):
bsum := w -> (1/4)*sum(b[n]*exp(-I*n*w*T1/4),n=-10..10):
plot([abs(asum(w)),abs(bsum(w))], w = -10..10, color = [red,blue]);
```

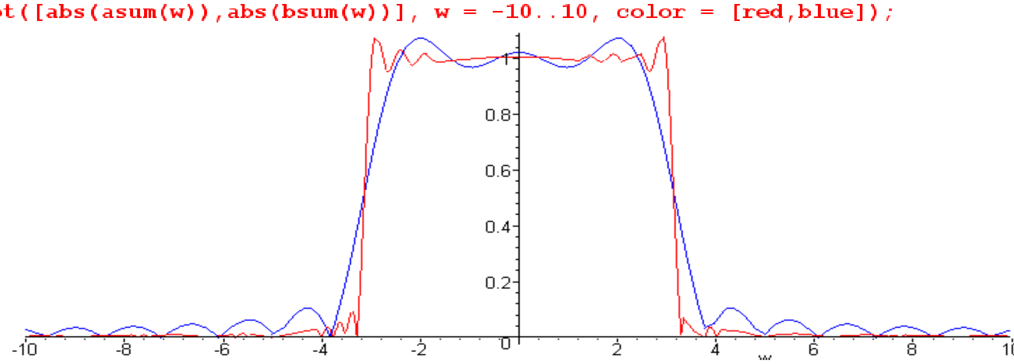


Fig 29.4

The 21-tap blue curve "brick wall", though not perfect, is pretty good, giving a fairly steep edge while maintaining a linear phase characteristic.

Below is the same plot with a wider range of ω (called w in the Maple code), showing the two nearest image spectra. Because we have selected $\Delta t = T_1/4 = 1/4$ (4x oversampling) for this filter, the first image spectrum on the right is centered at $4\omega_1 = 4(2\pi) \approx 25$.

```
plot([abs(asum(w)),abs(bsum(w))], w = -30..30, color = [red,blue]);
```

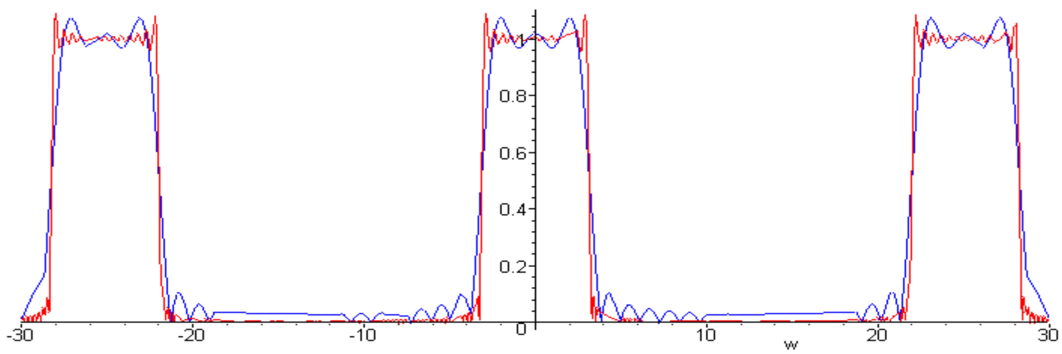


Fig 29.5

In this Section we have described a particular example of a low-pass filter. One problem that is evident from the plots above is that there is ringing in the spectra, known as "the Gibbs phenomenon". This can be reduced by multiplying the filter coefficients by a symmetric Gaussian-like weighting or "window" function. There are many proposed window functions associated with names Bartlett, Hann, Kaiser, Hamming, Blackman, etc.

30. Use of Oversampling in a D/A Converter Design

Here we discuss the design of an output circuit which contains a D/A converter. We are not concerned with the internal design of the actual D/A converter "chip" itself.

(a) A very simple D/A converter

Consider the following finite-length digital signal, which we assume has time spacing T_1,

y_n (n=-5..5) = { 1,2,3,3,2,1,1/5,-1,-2,-2,-1}

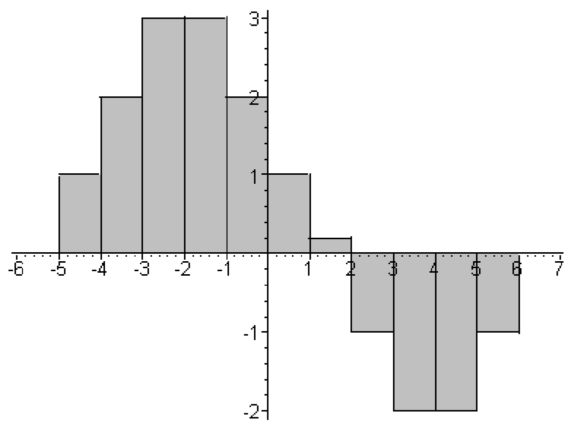


Fig 30.1

All samples other than those shown are 0.

Using an arbitrary pulse shape $x_{pulse}(t)$, we can construct an amplitude-modulated pulse train x(t) whose spectrum is X(ω), as shown in summary box (25.4),

$$x(t) = \sum_{n=-\infty}^{\infty} y_n x_{pulse}(t - t_n) \qquad t_n = n T_1 \qquad y_n = y(t_n) \tag{30.1}$$

$$X(\omega) = (1/T_1) X_{pulse}(\omega)\, Y'(\omega) , \tag{30.2}$$

where from summary box (23.5),

$$Y'(\omega) \equiv T_1 \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \quad . \qquad // \, Y''(z) = Y'(\omega)/T_1 \tag{30.3}$$

If we were to select $x_{pulse}(t)$ to be a box of height 1 and width T_1, then we could regard the stair-step outline function shown in Fig 30.1 as a candidate *analog* signal y(t) whose samples are the y_n. In this special case, y(t) = x(t) so Y(ω) = X(ω). From (9.2) and (12.1) we have

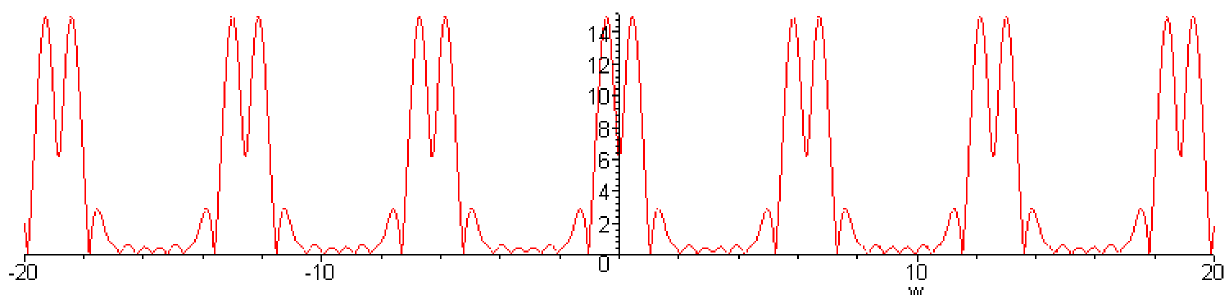
$$X_{pulse}(\omega) = X_{box}(\omega, T_1) = e^{-i\omega T_1/2}\, T_1 \, \text{sinc}(\omega T_1/2) \tag{30.4}$$

where the phase arises since the box $(0,T_1) = (0,1)$ is shifted $T_1/2$ to the right of the position of the symmetric box used in Section 9. Then using (30.2),

$$Y(\omega) = X(\omega) \;= e^{-i\omega T_1/2}\,\mathrm{sinc}(\omega T_1/2)\,Y'(\omega) = e^{-i\omega T_1/2}\,\mathrm{sinc}(\omega T_1/2)\,T_1 \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1}\,. \qquad (30.5)$$

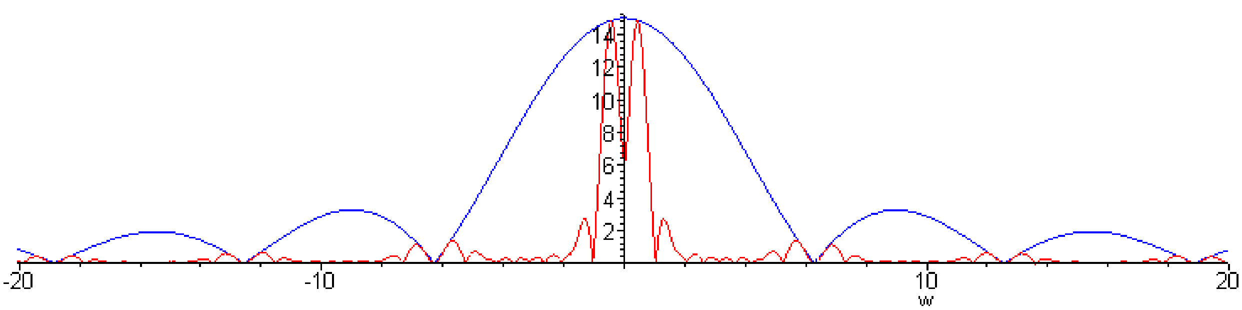
$Y(\omega)$ is the Fourier Integral Transform spectrum of the stair-step analog signal in Fig 30.1. Here are plots first of $|Y'(\omega)|$ from (30.3) using Fig 30.1 data, and second (red) of $|Y(\omega)|$ using (30.5).

```
y := array(-5..5,[1,2,3,3,2,1,.2,-1,-2,-2,-1]):
Yp := w -> sum(y[n]*exp(-I*n*w*T1),n=-5..5):
plot(abs(Yp(w)), w = -20..20, color = [red,blue],numpoints = 1200);
```



$|Y'(\omega)|$ Fig 30.2

```
sinc := x -> sin(x)/(x):
Y := w -> exp(-I*w*T1/2)* sinc(w*T1/2)*Yp(w):
plot([abs(Y(w)),abs(15*sinc(w*T1/2))], w = -20..20, color = [red,blue], numpoints =1200);
```



$|Y(\omega)|$ Fig 30.3

We see the expected image spectra in $Y'(\omega)$ in the first plot, but these spectra are quite suppressed in the second plot due to the taming effect of the sinc function zeros in (30.5), as shown in blue.

The above discussion describes the output of the following simple n-bit D/A converter design.

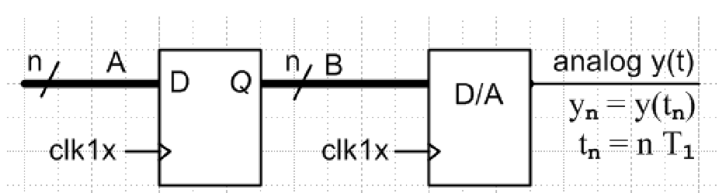


Fig 30.4

The purpose of the register on the left is to provide a stable signal on bus B to the D/A converter. We assume that the D/A converter is "glitch free" on its output, and just does what it should do.

(b) Oversampling just the D/A converter

We now trivially modify the above design by changing the D/A clock from clk1x to clk4x which runs 4X faster than clk1x,

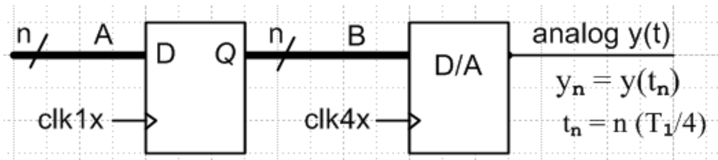


Fig 30.5

The D/A converter is now "4x oversampling" the data on the B bus. Besides making the D/A converter work harder, it seems clear that the output signal x(t) will be exactly the same as shown in Fig 30.1. Thus the plots of | Y'(ω) | and | Y(ω) | = | X(ω) | shown above apply to this design as well as that of Fig 30.4.

It is useful, nevertheless, to think of the output of the oversampled design as follows, where the nonvanishing y_n amplitudes are numbered n = -20 to + 23,

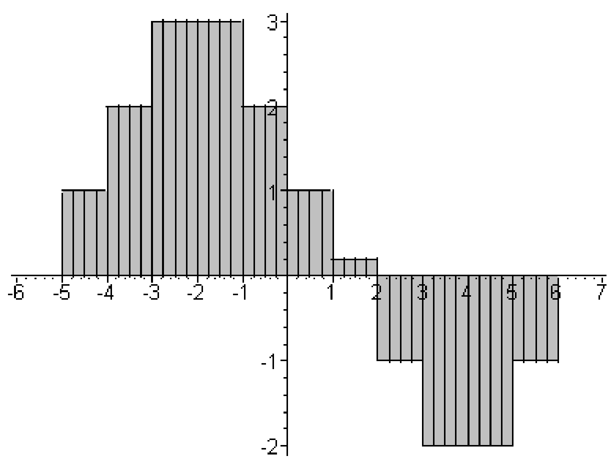


Fig 30.6

Now the output rectangles are 1/4 as wide because the D/A is clocking 4X faster. The analog outline is the same, but our analysis will be different. We shall now compute X(ω) in terms of the thin rectangles of Fig 30.6. Looking at the four y_0 = 1 samples to the right of the vertical axis, those four boxes will make this contribution to the spectrum

$$X(\omega) = X_{box}(\omega,T_1/4)\,[\ldots\ + y_0 + y_0\,e^{-i\omega(T_1/4)} + y_0\,e^{-2i\omega(T_1/4)} + y_0\,e^{-3i\omega(T_1/4)}\ + \ldots\,] \qquad (30.6)$$

Each thin box has a phase $e^{-i\omega(T_1/4)}$ relative to the box to its left due to (12.1). Since the pulse is 4x narrower than before, $X_{box}(\omega,T_1/4)$ is given by (30.4) with $T_1 \rightarrow T_1/4$.

We can write the square bracket in (30.6) as

$$[\, 1 + e^{-i\omega(T_1/4)} + e^{-2i\omega(T_1/4)} + e^{-3i\omega(T_1/4)} \,]\, y_0 \equiv F(\omega)\, y_0 \,. \qquad (30.7)$$

Every group of four terms will have this same common factor F(ω), so we can factor it out of the entire sum. The sum now looks like this:

$$X(\omega) = X_{pulse}(\omega,T_1/4)\, F(\omega)\, \{ \,..... \; y_0 + y_1 e^{-4i\omega(T_1/4)} + y_2 e^{-8i\omega(T_1/4)} + ... \} \qquad . \qquad (30.8)$$

But now the expression in curly brackets is exactly $Y'(\omega)/T_1$ of (30.3), our original Digital Fourier Transform spectrum of y(t). Thus we conclude that

$$X(\omega) = (1/T_1)\, X_{pulse}(\omega,T_1/4)\; F(\omega)\, Y'(\omega)$$

$$= [\, e^{-i\omega T_1/8}\, (1/4)\; \mathrm{sinc}(\omega T_1/8) \,]\, F(\omega)\, Y'(\omega) \quad . \qquad (30.9)$$

So this is X(ω) as computed in terms of the thin boxes of Fig 30.6. But we already argued that oversampling the D/A does not change the analog output signal x(t) or its spectrum X(ω), so somehow the expressions in (30.9) and (30.5) must be the same. This can only be true if

$$e^{-i\omega T_1/2}\, \mathrm{sinc}(\omega T_1/2) = [\, e^{-i\omega T_1/8}(1/4)\; \mathrm{sinc}(\omega T_1/8) \,]\, F(\omega) \qquad ?$$

or, writing out the sinc functions,

$$e^{-i\omega T_1/2}\, \sin(\omega T_1/2)(2/\omega T_1) = [\, e^{-i\omega T_1/8}(1/4)\; \sin(\omega T_1/8)\, (8/\omega T_1) \,]\, F(\omega) \quad ?$$

or

$$e^{-i\omega T_1/2}\, \sin(\omega T_1/2) = [\, e^{-i\omega T_1/8}\; \sin(\omega T_1/8) \,]\, F(\omega) \qquad ?$$

To verify this fact, we define $z \equiv e^{-i\omega T_1/4}$ (variable for the Z transform). The above then reads

$$z^2\, (z^{-2}-z^2) = [z^{1/2}\; (z^{-1/2} - z^{1/2})\,]\, [\, 1 + z + z^2 + z^3 \,] \qquad ?$$

or

$$(1-z^4) = (1 - z)(\, 1 + z + z^2 + z^3) \qquad ?$$

But this is a standard factorization so we find that X(ω) is indeed the same either way we compute it. For some other oversampling factor like 6x or 8x, the verification is similar. We have just shown that

$$X_{box}(\omega,T_1) = F(\omega)\, X_{box}(\omega,T_1/4)$$

which we can think of as saying the product of two filters on the right gives the one on the left.

(c) Add zero-stuffing to reduce aperture

We now add a multiplexor to our previous D/A converter design which causes the first sample in each group of four samples to pass through, but "grounds" the last three samples:

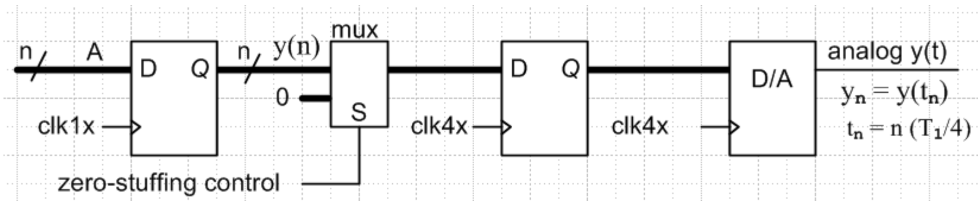


Fig 30.7

The output of this design is the following analog signal with pulse amplitudes y_n (where n = -20 to +23 as before, but three of every four samples in the region of interest are zero),

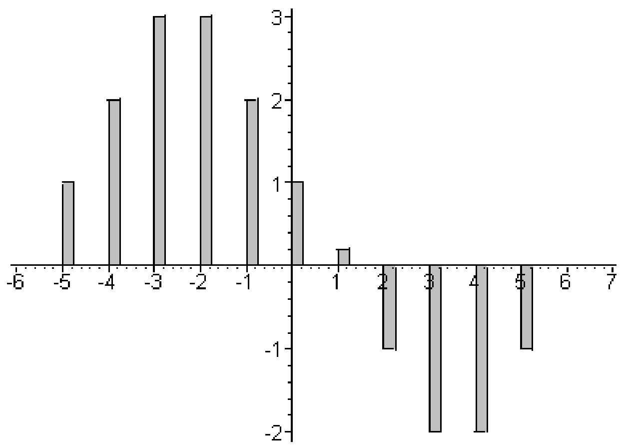


Fig 30.8

Due to the zero stuffing, we have in effect reduced the aperture of the signal from 100% to 25%. We saw in Section 26 how this broadens the sinc function envelope (narrower box ⇒ broader sinc) which in turn reduces the sinc distortion of the main spectrum. That same effect appears below.

Versions of equations (30.1,2,3,4) which apply to Figure 30.8 are (x_{pulse} is now the thin box)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n \, x_{pulse}(t - t_n) \qquad t_n = n\,(T_1/4) \qquad y_n = y(t_n) \tag{30.1}$$

$$X(\omega) = (4/T_1) X_{pulse}(\omega)\, Y'(\omega) , \tag{30.2}$$

$$Y'(\omega) \equiv (T_1/4) \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega n T_1/4} \quad . \tag{30.3}$$

$$X_{pulse}(\omega) = X_{box}(\omega, T_1/4) = e^{-i\omega T_1/8} \, (T_1/4) \, \text{sinc}(\omega T_1/8) \quad . \tag{30.4}$$

Combining the pieces gives,

$$X(\omega) = e^{-i\omega T_1/8} \, \text{sinc}(\omega T_1/8) \, [\, (T_1/4) \sum_{n=-\infty}^{\infty} y_n \, e^{-i\omega n T_1/4} \,]$$

$$= e^{-i\omega T_1/8} \, \text{sinc}(\omega T_1/8) \, [Y'(\omega)]$$

The plot of $|Y'(\omega)|$ is the same as shown in Fig 30.2 but with 1/4 the amplitude (the amplitudes y_n shown in Fig 30.8 are stored in array yz[n] as will be shown later),

```
Yp := w -> (T1/4)*sum(yz[n]*exp(-I*w*n*T1/4),n = -50..50):
plot(abs(Yp(w)), w = -20..20, color = red,numpoints = 1200);
```

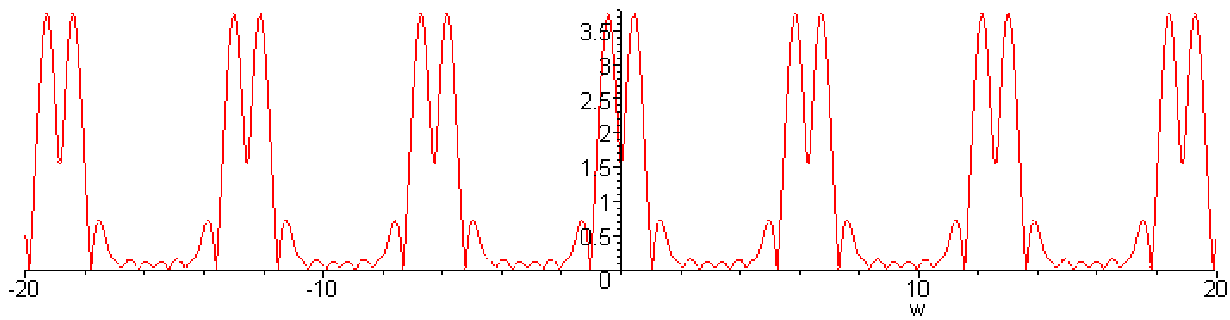


Fig 30.9

and the $X(\omega)$ plot is this,

```
X := w -> exp(-I*w*T1/8)*sinc(w*T1/8)*Yp(w):
plot([abs(X(w)),abs(3.75*sinc(w*T1/8))], w = -30..30, color = [red,blue],numpoints = 1200);
```

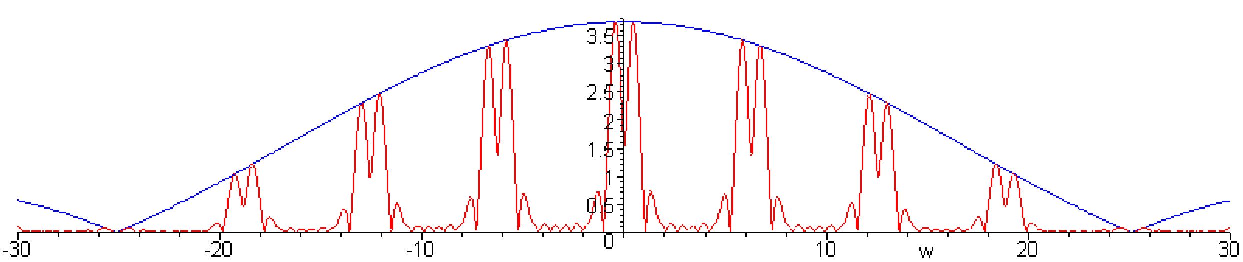


Fig 30.10

The good news is that the blue sinc distortion is smoother near the central main spectrum (compare to Fig 30.3). The bad news is that there are lots of high-amplitude image spectra the must be dealt with.

(d) Add an $\omega_1/2$ digital low-pass interpolation filter

The new D/A design is this,

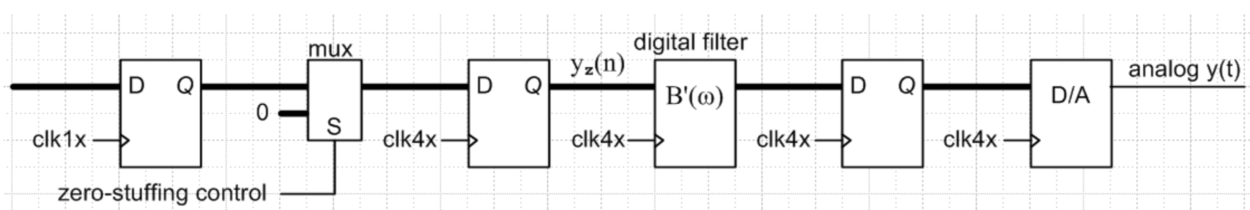


Fig 30.11

where $B'(\omega)$ is the transfer function of a low-pass filter which is clocked at the faster clk4x rate. We add low-cost registers at each stage in the pipeline to provide a stable input to the next stage.

In Section 29 we constructed an approximate brick-wall filter with this spectrum B'(ω),

```
wc := w1/2:  dt := T1/4:
for i from -10 to 10 do b[i] := evalf((Pi/wc)*sinc(wc*i*dt+.0000001)) od:
Bp := w -> (1/4)*sum(b[n]*exp(-I*n*w*T1/4),n=-10..10):
plot(abs(Bp(w)), w = -30..30);
```

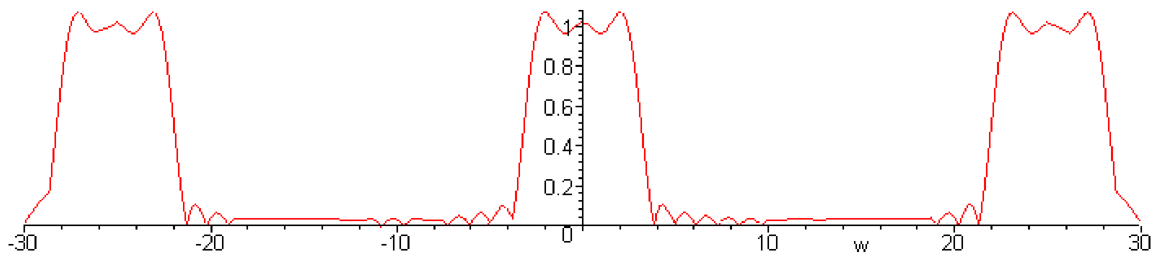


Fig 30.12

The output spectrum of the Fig 30.11 design which includes this filter is then

$X_{new}(\omega) = B(\omega)\, X(\omega)$,

where X(ω) was plotted in red in Fig 30.10. Here then is a plot of $X_{new}(\omega)$:

```
plot([abs(Bp(w)*X(w)),abs(3.75*sinc(w*T1/8))], w = -30..30,color = [red,blue]);
```

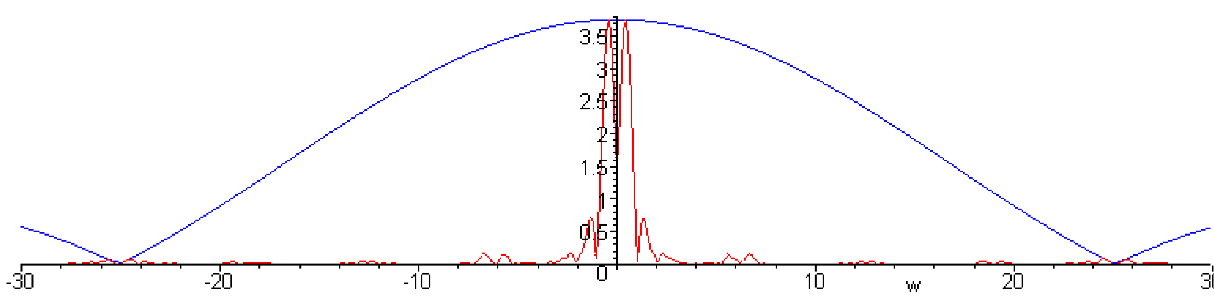


Fig 30.13

The effect of the digital filter is to remove the image spectra from Fig 30.10, a process sometimes called alias-rejection. Since this low-pass filter is not a perfect brick wall, there is some small distortion of the central spectrum. On the other hand, the aperture reduction due to oversampling with zero-stuffing has broadened the sinc hump perhaps alleviating the need for a sin(x)/x post-filter (Section 26). Residual high frequency data in the signal can be removed by a low-cost analog filter located to the right of the D/A converter in Fig 30.11.

Since we never specified the original signal y(t) for which Fig 30.1 is the sampled version, it is difficult to compare the spectrum of that y(t) with the output of the Fig 30.11 design. Nevertheless, plotting the time-domain output of the Figure 30.11 design is quite interesting.

It was noted that the thin bar amplitudes shown in Fig 30.8 (3/4 of which are 0) are stored in array yz[n]. Here is how that array is computed from the original amplitudes y[n] of Fig 30.1:

```
for i from -100 to 100 do yz[i] := 0 od:  #prefill with zeros
for i from -5*4 to 5*4+3 do
 if frac(i/4)=0 then yz[i] := y[floor(i/4)]
 else yz[i] := 0; fi;
od:
```

In terms of these yz amplitudes, the convolution sum in (29.5) which defines the action of our intepolation filter appears as

$$(y_{out})_n = (1/4) \sum_{m=-\infty}^{\infty} (y_z)_{n-m}\, b_m \qquad (29.5)$$

where $(y_{out})_n$ is the output of the filter which becomes, after two 4x clock delays, the output of the design of Fig 30.11.

Notice that the aperture duty cycle of 25% reduces the amplitude of the the filter output by 1/4, and there will be a corresponding reduction in the time-domain filter output signal. As aperture goes to 0, the entire signal eventually goes away. This fact, glossed over in Section 29, is one cost of using a small aperture. For comparison purposes, we shall omit the 1/4 factor in the following code which computes $(y_{out})_n$ for our 21 tap filter,

```
for i from -100 to 100 do b[i] := 0 od:   # prefill with all zeros
for i from -40 to 40 do b[i] := evalf((Pi/wc)*sinc(wc*i*dt+.0000001)) od:
for n from -20 to 20 do
  yout[n] := sum( yz[n-m]*b[m],m=-10..10):
od:
```

Here is a plot of the output sample values $y_{out}[n]$,

```
pointplot({seq([i,yout[i]],i=-40..40)},color = red,thickness = 2,symbol = box);
```

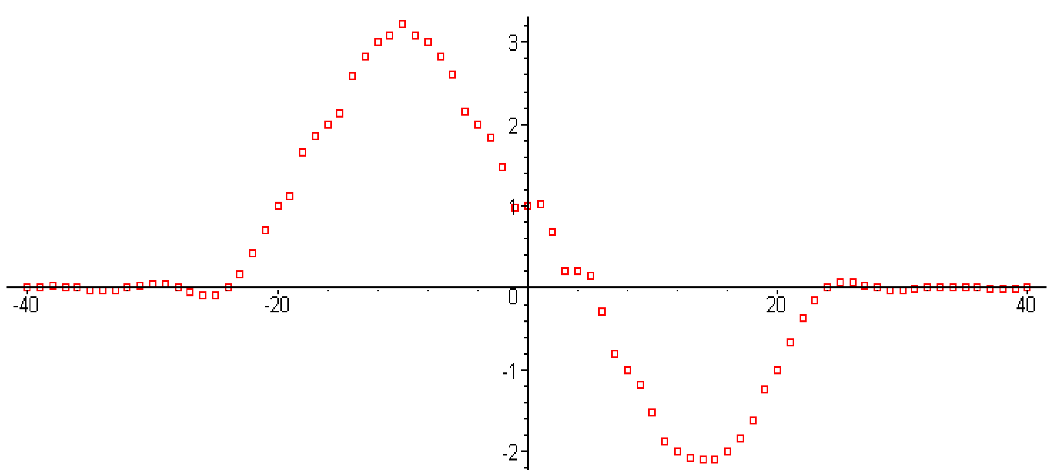


Fig 30.14

Since the D/A converter holds each sample for duration $T_1/4$, the actual analog output will have the following form, which we obtain using our ancient Maple V's primitive histogram routine (which we have been painfully using to make all the bar plots above),

```
for i from -40 to 40 do bar[i] := Weight(i/4..(i+1)/4,yout[i]/4) od:
histogram([seq(bar[n],n=-40..40)],color=gray);
```

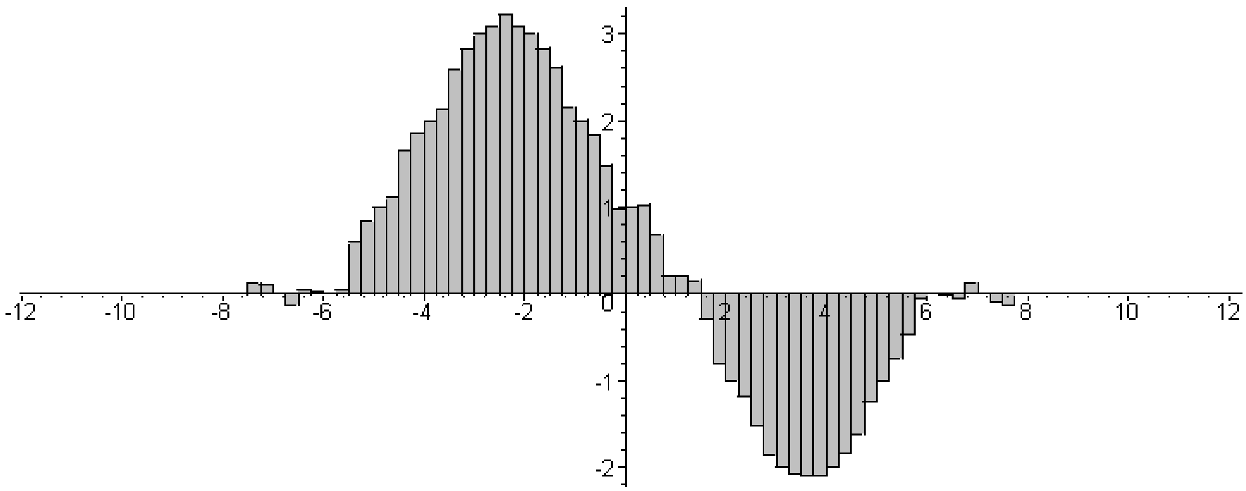


Fig 30.15

This may be compared with our starting digital signal of Fig 30.1,

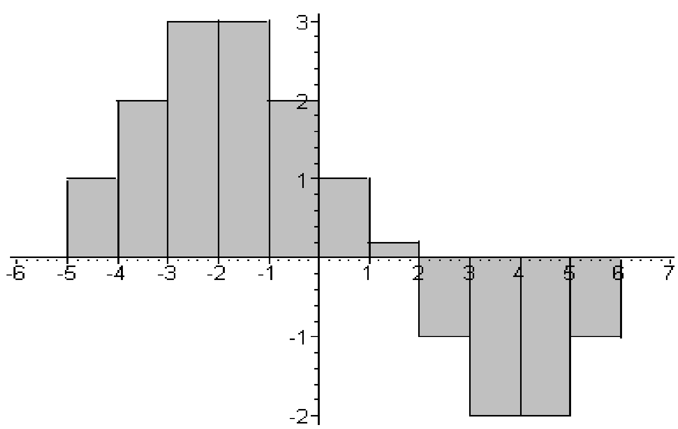


Fig 30.1

The output Fig 30.15 seems a little "ratty". If we increase the filter from 21 taps to 41 taps, things improve significantly, though there is still some ringing before and after the output pulse of interest (recall the comment about Gibbs phenomena at the end of Section 29).

```
for i from -100 to 100 do b[i] := 0 od:    # prefill with all zeros
for i from -20 to 20 do b[i] := evalf((Pi/wc)*sinc(wc*i*dt+.0000001)) od:
for n from -40 to 40 do
  yout[n] := sum( yz[n-m]*b[m],m=-20..20):
od:
```

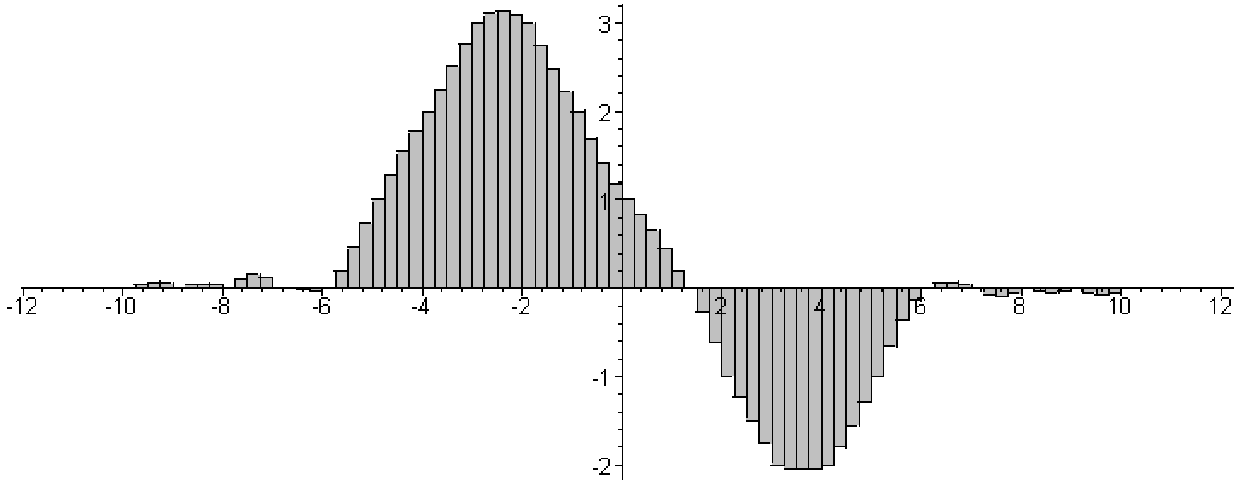


Fig 30.16

In the literature of oversampling, our oversampled digital low-pass filter is usually referred to as a digital interpolation filter, for obvious reasons comparing Fig 30.1 and Fig 30.16. In this application, since 3 out of every 4 incoming samples are zero from the zero-stuffing logic, it is possible to implement the filter more efficiently that we show in Fig 29.3 using polyphase techniques.

Chapter 5: Some Theoretical Topics

In this chapter we wander off on a more theoretical topic before returning in Chapter 6 to the practical computation of the power spectra of specific pulse train line codes. The material in this chapter is not used in that computation and the uninterested reader would do well to skip Section 31.

31. Spectral Dispersion Relations

(a) A simple integral equation for X(ω) analytic in the upper half plane

Let X(ω) be some arbitrary function of the complex variable ω which has two properties: (1) X(ω) is analytic in the upper half ω plane; (2) On any ray to ∞ in this upper half plane, X(ω) → X(∞), a constant which could be 0. Later we shall consider the case where "upper" ↔ "lower".

Consider then the following vanishing contour integral, where ω is real,

$$\int_C d\omega' \frac{X(\omega')}{\omega'-\omega} = 0 \ . \qquad (31.1)$$

Here, C is a counterclockwise contour which goes around the upper half ω' plane, but which detours infinitesimally around and above the pole at ω' = ω. The integral vanishes as usual since we can shrink the contour away to nothing.

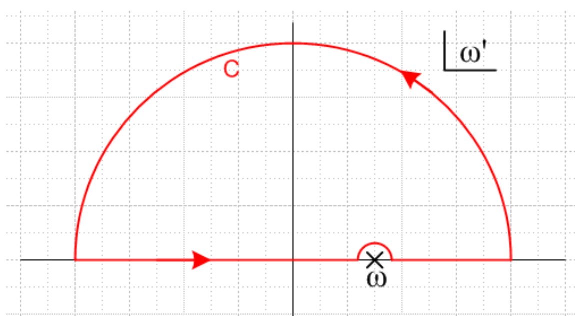


Fig 31.1

We can regard this integral as being made of three pieces:

(1) infinite semicircle. The contribution here is, using $\omega' = Re^{i\theta}$ and thinking R→ ∞,

$$\int_{SC} d\omega' \frac{X(\omega')}{\omega'-\omega} = \int_0^\pi (Re^{i\theta} i d\theta) \frac{X(Re^{i\theta})}{Re^{i\theta} - \omega} \approx i \int_0^\pi d\theta\, X(Re^{i\theta}) = X(\infty)\, i\pi$$

(2) tiny semicircular detour around the pole at ω = ω'. This gives minus one half the pole residue since the path goes half way around this pole the wrong way, so the contribution is – iπ X(ω) (see Appendix C text below Fig C.1, the partial residue rule).

(3) the two pieces (-∞,ω-ε) and (ω+ε,+∞) along the real ω' axis as ε→0. This is basically the integral along the real axis but missing the single point ω = ω'. As noted in Appendix C, this is called a Cauchy

principle part (principle value) integral, and sometimes people (including us) denote it with a little tick mark through the integral, $⨍$, while others use the notation P$\int$ or p.v.$\int$. The principle part integral is a limit just as are the previous two pieces of the contour C.

Thus, we can rewrite (31.1) as follows:

$$X(\infty)\, i\pi \;-\; i\pi\, X(\omega) + ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega} = 0$$

or

$$X(\omega) = X(\infty) + (1/i\pi)\, ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega} \tag{31.2}$$

where $X(\omega)$ is analytic in the upper half plane and in this half plane on any ray $X(\omega) \rightarrow X(\infty)$

This is an integral equation for $X(\omega)$ which involves a principle value integral of $X(\omega)$.

(b) A simple integral equation for X(ω) analytic in the lower half plane

We now repeat the previous section with upper → lower.

Let $X(\omega)$ be some arbitrary function of the complex variable ω which has two properties: (1) $X(\omega)$ is analytic in the lower half ω plane; (2) On any ray to ∞ in this lower half plane, $X(\omega) \rightarrow X(\infty)$, a constant which could be 0.

The new contour of interest is this

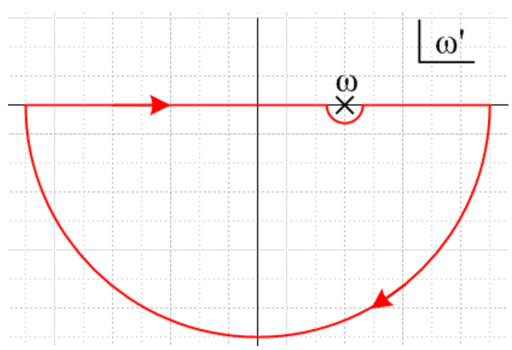


Fig 31.2

The differences now are that the great circle contour is now clockwise instead of counterclockwise, and that the little detour contour now goes "the right way" around the pole at $\omega' = \omega$. As a result, the contributions from both these terms get negated, but the principle part integral is unchanged. Equation (31.1) is as before but C is now this new contour

$$\int_C d\omega' \frac{X(\omega')}{\omega'-\omega} = 0 \; . \tag{31.1)'$$

so that

$$-X(\infty)\, i\pi \; + i\pi\, X(\omega) + \;⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega} \; = 0$$

or

$$X(\omega) = X(\infty) - (1/i\pi)\, ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega} \tag{31.3}$$

$X(\omega)$ is analytic in the lower half plane and in this half plane on any ray $X(\omega) \rightarrow X(\infty)$

Comparing to (31.2) one sees that the only difference is the sign of the integral. One way to understand this result is to think of reflecting the ω plane through the real axis, so that the imaginary axis is reflected and this is accounted for by taking $i \rightarrow -i$ in (31.2) to get (31.3). If we now define σ by

$$\sigma \equiv \begin{cases} +1 \text{ if } X(\omega) \text{ is analytic in the } \textit{upper} \text{ half plane with ray limit } X(\infty) \\ -1 \text{ if } X(\omega) \text{ is analytic in the } \textit{lower} \text{ half plane with ray limit } X(\infty) \end{cases} \tag{31.4}$$

We can combine our two results as follows :

$$X(\omega) = X(\infty) + \sigma\, (1/i\pi)\, ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega'-\omega} \; . \tag{31.5}$$

With our adopted phase sign convention $e^{-i\omega t}$ in the Fourier Integral Transform (1.1), we normally obtain spectral functions $X(\omega)$ which are analytic in the *lower* half plane and so $\sigma = -1$. For other sources (such as Stakgold) which use the reverse sign $e^{+i\omega t}$ of the Fourier transform phase, one would use $\sigma = +1$.

Example

We considered an RC filter in Section 4 (b) and found there the following transfer function,

$$G(\omega) = \frac{1}{1 + i\omega RC} = \frac{(-i/RC)}{\omega - i/RC} = \frac{(-i/\tau)}{\omega - i/\tau} \; . \qquad \tau \equiv RC \tag{31.6}$$

This function has a pole in the upper half plane at $\omega = i/RC$ and is analytic in the lower half plane with a ray limit there $G(\omega) \rightarrow 0$. For this function, (31.3) or (31.5) with $\sigma = -1$ claims that

$$G(\omega) = -\, (1/i\pi)\, ⨍_{-\infty}^{\infty} d\omega' \frac{G(\omega')}{\omega'-\omega} \tag{31.7}$$

or

$$\frac{1}{1 + i\omega RC} = -\,(1/i\pi)\, ⨍_{-\infty}^{\infty} d\omega' \frac{1}{\omega'-\omega} \frac{(-i/\tau)}{\omega' - i/\tau} \; = \; -\,(1/i\pi)(-i/\tau)\, ⨍_{-\infty}^{\infty} d\omega' \frac{1}{(\omega'-\omega)(\omega'-i/\tau)}$$

$$= \; (1/\pi\tau)\, ⨍_{-\infty}^{\infty} d\omega'' \frac{1}{\omega''(\omega'' + \omega - i/\tau)} \; = \; (1/\pi\tau)\, ⨍_{-\infty}^{\infty} dx \frac{1}{x(x + a)} \qquad a \equiv \omega - i/\tau, \;\; \tau = RC \; .$$

As a principle value integral exercise, we now evaluate the integral shown to verify that it really comes out being $G(\omega)$:

$$⨍_{-\infty}^{\infty} dx \frac{1}{x(x+a)} = \lim_{\varepsilon\to 0} \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right] \frac{dx}{x(x+a)}$$

$$= (1/a) \lim_{\varepsilon\to 0} \{ \ln(\frac{x}{x+a})|^{-\varepsilon}_{-\infty} + \ln(\frac{x}{x+a})|^{\infty}_{\varepsilon}]$$

$$= (1/a) \lim_{\varepsilon\to 0} \{ \ln(\frac{-\varepsilon}{-\varepsilon+a}) - \ln(1) + \ln(1) - \ln(\frac{\varepsilon}{\varepsilon+a}) \} = (1/a) \lim_{\varepsilon\to 0} \{ \ln(\frac{-\varepsilon}{-\varepsilon+a}) - \ln(\frac{\varepsilon}{\varepsilon+a}) \}$$

$$= (1/a) \lim_{\varepsilon\to 0} \{ \ln(\frac{-\varepsilon}{-\varepsilon+a}\frac{\varepsilon+a}{\varepsilon}) \} = (1/a) \lim_{\varepsilon\to 0} \{ \ln(\frac{\varepsilon+a}{\varepsilon-a}) \} = (1/a) \lim_{\varepsilon\to 0} \{ \ln(-1-i\varepsilon') \}$$

$$= (1/a) \lim_{\varepsilon\to 0} \{ \ln(e^{-i\pi}) \} = (1/a)(-i\pi) \ .$$

The term $-i\varepsilon'$ represents the fact that the log argument has a small negative imaginary part, which takes a bit of work to show:

$$\frac{\varepsilon+a}{\varepsilon-a} = -\frac{a+\varepsilon}{a-\varepsilon}\frac{a^*-\varepsilon}{a^*-\varepsilon} = \frac{-|a|^2 + 2i\varepsilon Im(a) - \varepsilon^2}{|a|^2 - 2\varepsilon Re(a) + \varepsilon^2} = \frac{-|a|^2 + 2i\varepsilon(-1/\tau) - \varepsilon^2}{|a|^2 - 2\varepsilon\omega + \varepsilon^2} \ .$$

We have then shown that the right side of (31.7) evaluates to the left side,

$$-(1/i\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{G(\omega')}{\omega'-\omega} = (1/\pi\tau) ⨍_{-\infty}^{\infty} dx \frac{1}{x(x+a)} = (1/\pi\tau)(1/a)(-i\pi) = -\frac{i/\tau}{a} = \frac{-i/\tau}{\omega-i/\tau} = G(\omega) \ .$$

(c) Dispersion Relations for X(ω)

Notice in (31.5) the very important factor of (1/i). If we now break X(ω) into its real and imaginary parts and then write down the real and imaginary parts of equation (31.5), we find that Re(X) and Im(X) are related to each other by the following two equations:

$$Re[X(\omega)] = Re[X(\infty)] + \sigma(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{Im[X(\omega')]}{\omega'-\omega} \qquad (31.8a)$$

$$Im[X(\omega)] = Im[X(\infty)] - \sigma(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{Re[X(\omega')]}{\omega'-\omega} \ . \qquad (31.8b)$$

Basically, this says that the real part of X(ω) along the real axis completely determines the imaginary part, and vice versa. One cannot arbitrarily set the real and imaginary parts independently. This is a general fact about analytic functions X(ω).

Next, let us assume in addition that X(ω) is the spectrum of a *real* function x(t). As we saw in Section 7, this implies the reflection rule X(–ω) = X(ω)*, which says X(ω) is Hermitian. Thus, we can fold the negative portions of the above integrations over to the positive side. First (we omit principle value tick marks just for a while),

$$\int_{-\infty}^{0} d\omega' \frac{Im[X(\omega')]}{\omega'-\omega} = \int_{+\infty}^{0} [-d\omega''] \frac{Im[X(-\omega'')]}{-\omega''-\omega} = \int_{+\infty}^{0} [-d\omega''] \frac{Im[X(\omega'')^*]}{-\omega''-\omega}$$

$$= \int_{+\infty}^{0} [-d\omega''] \frac{-Im[X(\omega'')]}{-\omega''-\omega} = \int_{0}^{\infty} [d\omega''] \frac{-Im[X(\omega'')]}{-\omega''-\omega} = \int_{0}^{\infty} d\omega'' \frac{Im[X(\omega'')]}{\omega''+\omega}$$

and therefore

$$\int_{-\infty}^{\infty} d\omega' \frac{Im[X(\omega')]}{\omega'-\omega} = \int_{0}^{\infty} d\omega' \frac{Im[X(\omega')]}{\omega'+\omega} + \int_{0}^{\infty} d\omega' \frac{Im[X(\omega')]}{\omega'-\omega}$$

$$= \int_{0}^{\infty} d\omega' \, Im(X(\omega')) \left[\frac{1}{\omega'+\omega} + \frac{1}{\omega'-\omega}\right] = 2 \int_{0}^{\infty} d\omega' \frac{\omega' Im[X(\omega')]}{\omega'^2-\omega^2} \quad .$$

The other integral can be folded in a similar manner,

$$\int_{-\infty}^{0} d\omega' \frac{Re[X(\omega')]}{\omega'-\omega} = \;....\; = - \int_{0}^{\infty} d\omega' \frac{Re[X(\omega')]}{\omega'+\omega}$$

$$\int_{-\infty}^{\infty} d\omega' \frac{Re[X(\omega')]}{\omega'-\omega} = \int_{0}^{\infty} d\omega' \, Re(X(\omega')) \left[- \frac{1}{\omega'+\omega} + \frac{1}{\omega'-\omega}\right] = 2\omega \int_{0}^{\infty} d\omega' \frac{Re[X(\omega')]}{\omega'^2-\omega^2}$$

We then rewrite (31.5) as , valid for $X(-\omega) = X(\omega)^*$ (tick marks restored),

$$Re[X(\omega)] = Re[X(\infty)] + \sigma\, (2/\pi) ⨍_{0}^{\infty} d\omega'\omega' \frac{Im[X(\omega')]}{\omega'^2-\omega^2} \tag{31.9a}$$

$$Im[X(\omega)] = Im[X(\infty)] - \sigma(2/\pi) ⨍_{0}^{\infty} d\omega' \frac{Re[X(\omega')]}{\omega'^2-\omega^2} \quad . \tag{31.9b}$$

These two equations are completely general, given the assumptions we have made. They are associated with the names **Kramers and Kronig** who wrote similar equations in 1926 for certain functions connected with the index of refraction and the dispersion of light (hence "dispersion relations").

(d) Dispersion Relations for γ(ω)

In filter theory, one thinks of X(ω) as the "transfer function" of a filter. It is usually easier to think in terms of the function γ(ω) which we define as

$$\gamma(\omega) \equiv - \ln [X(\omega)] = \alpha(\omega) + i\, \beta(\omega) \quad . \tag{31.10}$$

Then we get

$$X(\omega) = e^{-\gamma(\omega)} = e^{-\alpha(\omega)}\, e^{-i\beta(\omega)} \quad . \tag{31.11}$$

Notice that we defined $\gamma(\omega)$ with a minus sign, so both exponents have minus signs. The real quantities $\alpha(\omega)$ and $\beta(\omega)$ are the attenuation and phase functions of the filter.

Can we apply the dispersion relations to the function $\gamma(\omega)$ instead of $X(\omega)$? Yes, provided $\gamma(\omega)$ meets the same requirements assumed for $X(\omega)$. If $X(\omega)$ has a pole in the upper half ω plane, as in (31.6), then $\gamma(\omega)$ has a branch cut singularity in the upper half plane starting at the pole and going off to the left. No problem since $\gamma(\omega)$ is still analytic in the lower half plane. However, consider:

$$\gamma(\omega) \equiv -\ln[X(\omega)] = +\ln[1/X(\omega)] \ .$$

This says that a zero in $X(\omega)$ is just as bad as a pole from $\gamma(\omega)$'s point of view. A zero of $X(\omega)$ in the lower half plane means $\gamma(\omega)$ has a branch cut in the lower half plane starting at this zero location and going off to the left, and this invalidates our conditions ($\sigma = -1$).

Thus, we must now assume that $X(\omega)$ has neither zeros nor poles in the lower half plane, which maps into the Z Transform H"(z) having neither zeros nor poles outside the unit circle in Fig 24.1. A filter satisfying this condition is called a **minimum phase filter**.

Since we have assumed $X(\omega)$ goes to $X(\infty)$ on the great circle at infinity, we know that $\gamma(\omega)$ goes to $\gamma(\infty) = -\ln[X(\infty)]$, so no extra assumption is needed here. If $X(\infty) = 0$, then $\gamma(\infty) = -\infty$, which is a little inconvenient. It just says that the attenuation of our filter is infinite as $\omega \rightarrow \infty$.

So now we can write the dispersion relations analogous to (31.8) for $\gamma(\omega)$ instead of $X(\omega)$, assuming now that $X(\omega)$ has neither poles nor zeros in the lower half ω plane. We must choose $\sigma = -1$ in (31.4), and note that $\mathrm{Re}[\gamma(\omega)] = \alpha(\omega)$ and $\mathrm{Im}[\gamma(\omega)] = \beta(\omega)$:

$$\alpha(\omega) = \alpha(\infty) + \sigma\,(1/\pi) \,⨍_{-\infty}^{\infty} d\omega' \frac{\beta(\omega')}{\omega'-\omega} \tag{31.12a}$$

$$\beta(\omega) = \beta(\infty) - \sigma\,(1/\pi) \,⨍_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega} \tag{31.12b}$$

Now if x(t) is real so that $X(\omega) = X(-\omega)^*$, we find that $\gamma(\omega)$ is *also* Hermitian since $X(\omega) = e^{-\gamma(\omega)}$:

$$X(-\omega) = X(\omega)^* \qquad \Rightarrow \qquad \gamma(-\omega) = \gamma(\omega)^* \tag{31.13}$$

$$\Rightarrow \qquad \alpha(-\omega) = \alpha(\omega) \quad \text{and} \quad \beta(-\omega) = -\,\beta(\omega)$$

Since $\gamma(\omega)$ is Hermitian, we can process (31.12) just as we did processed (31.8) to get

$$\alpha(\omega) = \alpha(\infty) + \sigma\,(2/\pi) \,⨍_{0}^{\infty} d\omega'\omega' \frac{\beta(\omega')}{\omega'^2-\omega^2} \tag{31.14a}$$

$$\beta(\omega) = \beta(\infty) - \sigma\,(2/\pi)\ \omega \,⨍_{0}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'^2-\omega^2} \tag{31.14b}$$

The main point of the above is that the phase of a minimum phase filter is completely determined by its attenuation, and vice versa. Even for general filters there will be some relation like the above, but it will include terms to describe the zeros of X(ω) in the lower half plane. The conclusion that the phase and attenuation cannot be independently set is unavoidable.

(e) Dispersion and Attenuation

We showed in (21.19) that the group delay of a filter is given by

$$\tau_d = d\varphi/d\omega \qquad \text{where} \qquad B(\omega) = |B(\omega)|\, e^{-i\varphi(\omega)} \;.$$

Translating that into our current context, we get

$$\tau(\omega) = d\beta(\omega)/d\omega \qquad \text{where} \qquad X(\omega) = e^{-\gamma(\omega)} = e^{-\alpha(\omega)}\, e^{-i\beta(\omega)} \;. \tag{31.15}$$

If we define the integral appearing in (31.14b) as K(ω), including the (2/π),

$$K(\omega) \equiv (2/\pi) \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2-\omega^2} \tag{31.16}$$

then we find that

$$\tau(\omega) = d\beta(\omega)/d\omega = d\,[\,\omega\, K(\omega)\,]\,/d\omega \;.$$

If the integral K(ω) were somehow a constant κ, we would conclude that τ(ω) = κ, and the filter would be "non-dispersive". As discussed in Section 21 (b), this means the filter is "linear phase" and the group delay is a constant independent of ω. All frequency components of a pulse packet would then traverse the filter in the same time, so the pulse does not spread out (disperse) in time.

Obviously K(ω) cannot really be independent of ω, so a non-dispersive minimum phase filter does not exist. However, over certain ranges of ω where K(ω) is very slowly varying, such a filter can be reasonably non-dispersive. This would be a region of ω far away from any region where the attenuation α(ω) strongly varies.

Crude Proof: Suppose α(ω) is very smoothly varying near some ω. In the integral (31.16) for K(ω) near ω we expect the main contribution to come from ω' close to ω, (ω-a,ω+a) for small a, since the denominator is 0 and therefore amplifies the numerator there. The denominator is roughly 2ω(ω'- ω) which is a pole. If we assume that α(ω) has some linear form α(ω') ≈ α(ω) + (ω-ω')α'(ω) near ω, then this integration region which would normally be highly amplifying in fact yields the following,

$$K(\omega) \equiv (2/\pi) \int_0^\infty d\omega' \frac{\alpha(\omega')}{\omega'^2-\omega^2} \approx (2/\pi) \int_{\omega-a}^{\omega+a} d\omega' \frac{\alpha(\omega) + (\omega-\omega')\alpha'(\omega)}{\omega'^2-\omega^2}$$

$$\approx (2/\pi)\, \alpha(\omega) \frac{1}{2\omega} \left\{ \int_{\omega-a}^{\omega+a} \frac{d\omega'}{\omega-\omega'} \right\} + (2/\pi)\, \alpha'(\omega) \frac{1}{2\omega} \left\{ \int_{\omega-a}^{\omega+a} \frac{(\omega-\omega')d\omega'}{\omega-\omega'} \right\}$$

$$= (2/\pi)\, \alpha(\omega) \frac{1}{2\omega} \;\{ 0 \}\; + (2/\pi)\, \alpha'(\omega) \frac{1}{2\omega}\, 2a \;\approx\; (2a/\pi) \frac{\alpha'(\omega)}{\omega} \;\approx\; 0 \;\; \text{since } \alpha'(\omega) \text{ is small}$$

The rest of the integration region which is far from ω yields a contribution to K(ω) which is weakly dependent on ω and which we can regard as roughly constant K(ω) ≈ κ for some band Δω. We then get our desired approximate linear phase and constant group delay, and therefore very small dispersion,

$$\tau(\omega) = d\beta/d\omega = d\,[\,\omega K(\omega)\,]/d\omega \approx d\,[\,\omega \kappa\,]/d\omega = \kappa \;.$$

However, if α(ω) varies significantly near ω, our linear term with α'(ω) might be large and there will likely be additional higher terms in the Taylor expansion of α(ω) so K(ω) might then vary strongly with ω due to the integration contribution from region (ω-a,ω+a). In this case, we get non-linear phase and dispersion.

To repeat the claim above: For ω far from regions where attenuation α(ω) significantly varies, we expect K(ω) to be roughly constant and so we have nearly linear phase, nearly constant group velocity, and small dispersion.

Attenuation and dispersion are intertwined. You can't have one without the other. This is a general fact one learns from the dispersion relations, without any specific filter in mind.

(f) Application to coaxial cable

Consider an infinitely long coaxial cable driven at its left end at z = 0. A coaxial cable acts as a filter G(ω). In the frequency domain, if we drive the cable with I(ω), the output O(ω,z) at z is

$$O(\omega, z) = G(\omega, z)\, I(\omega) \qquad G(\omega) = e^{-\gamma(\omega) z} \qquad \gamma(\omega) = \alpha(\omega) + i\beta(\omega) \;. \tag{31.17}$$

For a coaxial cable one has

$$\gamma = \sqrt{(R+i\omega L)(G+i\omega C)} \tag{31.18}$$

where R,L,G and C are resistance, inductance, conductance (across the dielectric) and capacitance all per unit length of the cable. If we ignore ohmic losses in both the conductor and the dielectric, then R = G = 0 and we get

$$\gamma = i\omega\sqrt{LC} \;.$$

The inductance L is generally independent of ω, but $C = \varepsilon(\omega)2\pi\varepsilon_0/\ln(b/a) = k_1\, \varepsilon(\omega)$, where ε(ω) is the dielectric "constant", which in general is not constant as a function of ω. Thus we have

$$\gamma(\omega) = i\omega\sqrt{Lk_1}\,\sqrt{\varepsilon(\omega)} \;.$$

From Maxwell's equations one knows that the index of refraction of a medium is given by

$$n(\omega) = \sqrt{\mu\varepsilon} = \sqrt{\varepsilon(\omega)}\ /k_2 \ .$$

So we then have,

$$\gamma(\omega) = i\omega\sqrt{Lk_1}\, k_2\, n(\omega) = i\omega k_3\, n(\omega)$$

$$= i\omega k_3[\ \mathrm{Re}(n) + i\ \mathrm{Im}(n)\ ] = -\ \omega k_3 \mathrm{Im}(n) + i\omega k_3 \mathrm{Re}(n) = \alpha(\omega) + i\beta(\omega)$$

so

$$\alpha(\omega) = -\omega k_3 \mathrm{Im}[n(\omega)] \qquad\qquad \beta(\omega) = \omega k_3 \mathrm{Re}[n(\omega)] \ .$$

If it were true that Re[n(ω)] were independent of ω, then β(ω) would have linear phase and we would then have constant group delay and no dispersion in the coaxial cable.

It turns out that the index n(ω) has the right properties for the dispersion relations (31.8) to be valid, so

$$\mathrm{Re}[n(\omega)] = \mathrm{Re}[n(\infty)] + \sigma(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}[n(\omega')]}{\omega'-\omega} \qquad (31.19a)$$

$$\mathrm{Im}[n(\omega)] = \mathrm{Im}[n(\infty)] - \sigma(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}[n(\omega')]}{\omega'-\omega} \qquad (31.19b)$$

where usually Re[n(∞)] = 1 and Im[n(∞)] = 0. If it happened that Im[n(ω)] were very small, then (31.19a) says that Re[n(ω)] = Re[n(∞)] = constant, just what we want to get no cable dispersion.

In a non-polar dielectric, like polyethylene or teflon, n(ω) does in fact have a very small imaginary part for frequencies below the electromagnetic resonances of the medium. Thus, if we could ignore ohmic losses in the conductors, coaxial cables using these materials as dielectrics would be non-dispersive up to infrared frequencies -- where vibrational and rotational resonances set in.

Dispersion relations are often written for other functions such as the dielectric constant ε(ω). In this case one can regard the relationship between electric displacement D and electric field E

$$D(\omega) = e(\omega)\ E(\omega) \qquad (31.20)$$

as a "filter", where everything is evaluated at the same point in space.

(g) The Dispersion Relation expressed in terms of the Hilbert Transform

Recall the integral equation which is the starting point for the dispersion relation discussion above,

$$X(\omega) = X(\infty) + i\ \sigma\ (1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega-\omega'} \qquad \mathit{(31.5)}$$

$$\sigma \equiv \begin{cases} +1 \text{ if } X(\omega) \text{ is analytic in the } \textit{upper} \text{ half plane with ray limit } X(\infty) \\ -1 \text{ if } X(\omega) \text{ is analytic in the } \textit{lower} \text{ half plane with ray limit } X(\infty) \end{cases} \qquad (31.4)$$

where we have introduced some offsetting minus signs. The integral appearing here is in fact another transform in our growing cornucopia of transforms, this one being the **Hilbert Transform**,

$$X_h(\omega) \equiv (1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega-\omega'} \quad . \qquad \text{// projection = transform} \qquad (31.21)$$

Thus, it happens that the dispersion relation (31.5) can be written

$$X(\omega) = X(\infty) + i\,\sigma\, X_h(\omega) \quad . \qquad (31.22)$$

If $X(\infty) = 0$ and $\sigma = +1$, any solution to the dispersion relation (31.5) must be a function $X(\omega)$ which is equal to its own Hilbert transform times $i\sigma$. In Appendix C we find that (with $\beta = 1$)

$$X(\omega) = e^{i\omega} \qquad \Leftrightarrow \qquad X_h(\omega) = -i\, e^{i\omega} \quad . \qquad (C.43)$$

This $X(\omega)$ is then a particular solution to the dispersion relation since $i\, X_h(\omega) = e^{i\omega} = X(\omega)$ and in the upper half plane $X(\infty) \sim e^{i(+i\infty)} = e^{-\infty} = 0$. In (31.7) we found another solution for the case $\sigma = -1$ where $X(\infty) = 0$ in the lower half plane.

The general study of the integral equation (31.5)

$$X(\omega) = X(\infty) + \sigma\,(1/i\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{1}{\omega'-\omega} X(\omega') \qquad (31.5)$$

and its possible solutions is complicated because the kernel $1/(\omega'-\omega)$ (called in this case a Cauchy kernel) is singular (it blows up when $\omega = \omega'$). See for example Polyanin and Manzhirov Chapter 15. It is more useful to think of the dispersion relation as an integral condition which a spectrum $X(\omega)$ must satisfy rather than an integral equation which completely determines that spectrum.

Chapter 6: Power in Pulse Trains

32. The Autocorrelation Function

Here we deal with some preliminary matters before studying the power spectra of pulse trains.

Start with a reasonable function x(t). Define the autocorrelation function of x(t) as follows:

$$r_{\mathbf{x}}(t) \equiv \int_{-\infty}^{\infty} dt' \, x(t')^* \, x(t' + t) \quad . \tag{32.1}$$

The integrand is the function evaluated at time t' times the same function evaluated at later time t'+t.
Some sources define $r_{\mathbf{x}}(t)$ with an extra overall "normalizing" constant factor. For example, if one were to define $a_{\mathbf{x}}(t) \equiv r_{\mathbf{x}}(t)/ T$ where T is the duration of a pulse train, then $a_{\mathbf{x}}(t)$ and $r_{\mathbf{x}}(t)$ have different dimensions. Below we show that our $r_{\mathbf{x}}(t)$ has dimensions of energy, so $a_{\mathbf{x}}(t)$ would have dimensions of power. We prefer defining autocorrelation as shown in (32.1) with no normalizing constant.

A simple reflection property follows from the above definition (use t" = t' + t):

$$r_{\mathbf{x}}(-t) = r_{\mathbf{x}}(t)^* \quad . \tag{32.2}$$

For real x(t) then $r_{\mathbf{x}}(t)$ is an even function of t, in (32.1) we could put either +t or -t in the last parentheses.

Although we have not yet mentioned statistics and randomness, one could easily imagine the following situation. Suppose x(t') is some sort of random function ("noise") that takes values in the range -1 to 1. It seems likely that for a value of the separation t that is larger than some small value, one might get $r_{\mathbf{x}}(t) = 0$. The vague argument would be that there is no "correlation" between x(t') and x(t'+t), so the product of these two functions ought to be pretty random, and a sum of random numbers in the range -1 to 1 ought to be zero. Even in this case, we can see that the result is not zero if t = 0, since we are then summing a positive quantity. In fact, $r_{\mathbf{x}}(0)$ is the area under $x(t)^2$.

(a) Autocorrelation function for a Square Pulse

Before going any further, let us compute the autocorrelation function for some simple case we are familiar with. A good candidate is x(t) = a square pulse of width τ and amplitude A. As in (9.1),

$$x_{\mathbf{pulse}}(t) = A \left[\theta(t + \tau/2) - \theta(t - \tau/2) \right] \quad . \tag{9.1}$$

One can easily do the above integral (32.1) to get the answer, but it is very obvious what the answer is. We are multiplying a box times a box shifted by t. Where they overlap, the integrand is A^2. The boxes only overlap if the absolute value of shift t is less than the width τ of the pulse. If this is so, the size of the overlap is τ - |t|. If |t| is larger than τ, there is no overlap, so the integral is 0. Thus,

$$r_{\mathbf{pulse}}(t) = A^2(\tau - |t|) \, \theta(\tau - |t|) = (A^2\tau) \left[1 - |t|/\tau \right] \theta(\tau - |t|) \tag{32.3}$$

and we find that the autocorrelation function is a triangle whose base is twice the pulse width,

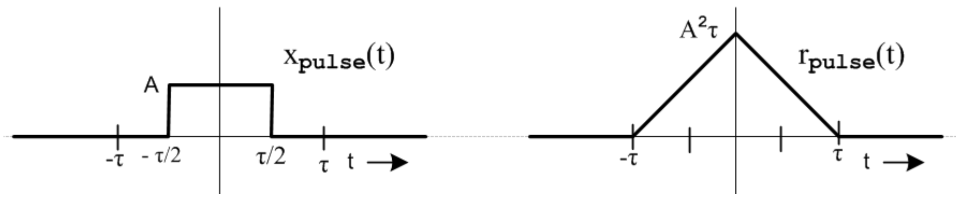


Fig 32.1

In general, if x(t) has some finite width τ, $r_x(t)$ will have width 2τ.

(b) Energy, power and spectral energy density for a finite signal x(t)

The total energy in a finite duration signal x(t) can be computed in either the t-domain or the ω-domain using Parseval's formula (10.5), to which we add 1/R to each side,

$$E = \int_{-\infty}^{\infty} dt\, |x(t)|^2/R = \int_{-\infty}^{\infty} d\omega\, \frac{|X(\omega)|^2}{2\pi R} \quad . \tag{32.4}$$

If we think of x(t) as the voltage across a resistor R, then dt $x^2(t)/R$ is the energy delivered to the resistor in time dt, and the integral on the left is the total energy in signal x(t). The dimensions on the right are, looking at (1.1),

$$d\omega \frac{|X(\omega)|^2}{2\pi R} = \text{sec}^{-1}\,(\text{volt-sec})^2/\text{ohms} = (\text{volt}^2/\text{ohms})\text{sec} = \text{watts-sec} = \text{joules} = \text{energy} \ . \tag{32.5}$$

Setting R = 1Ω, we can write this as

$$E = \int_{-\infty}^{\infty} dt\, p(t) = \int_{-\infty}^{\infty} d\omega\, \mathcal{E}(\omega) \tag{32.6}$$

$p(t) \equiv |x(t)|^2$ = energy density in the t-domain (joule/sec = watt) (= instantaneous power)
p(t)dt = energy in dt (joules)

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec)
$\mathcal{E}(\omega)d\omega$ = energy in dω (joules)

We shall refer to $\mathcal{E}(\omega)$ as the **spectral energy density** of signal x(t) whose Fourier Transform is X(ω). The isolated single pulse $x_{pulse}(t)$ has a corresponding $\mathcal{E}_{pulse}(\omega)$. The function p(t) is the "temporal energy density".

(c) The Wiener-Khintchine relation

Changing to t" = -t', we can trivially rewrite the definition (32.1) as follows:

$$r_x(t) = \int_{-\infty}^{\infty} dt''\, x(t - t'')\, x(-t'')^* \ . \qquad \text{// energy units} \tag{32.7}$$

From now on, we assume x(t) is a **real** valued function. Recall now the convolution theorem (3.6),

$$a(t) = \int_{-\infty}^{\infty} dt'' \, b(t-t'') \, c(t'') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega) \, C(\omega) \; . \qquad \textit{(3.6)}$$

We see that (32.7) has the standard convolution equation form where we select

$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$
$c(t) = x(-t)^* \quad \leftrightarrow \quad C(\omega) = X(\omega)^* \quad$ // from (7.2)

Thus, the diagonalized frequency domain form $A(\omega) = B(\omega)\, C(\omega)$ is

$$R_x(\omega) = |X(\omega)|^2 \; . \qquad (32.8)$$

Dividing by 2π we find that

$$\mathcal{E}(\omega) = (1/2\pi) \, R_x(\omega) \; . \qquad (32.9)$$

This says that the spectral energy density $\mathcal{E}(\omega)$ of signal x(t) is $1/2\pi$ times the Fourier Integral Transform $R_x(\omega)$ of the autocorrelation function $r_x(t)$ of the signal x(t). This result is sometimes called the Wiener-Khintchine [Khintchin] theorem.

Note also from (32.1) that $r_x(t)$ evaluated at t = 0 gives the total energy in signal x(t),

$$r_x(0) \;=\; \int_{-\infty}^{\infty} dt \, |x(t)|^2 \equiv \; E = \text{total energy in signal } x(t) \qquad (32.10)$$

For us, the significance of (32.9) is that we can "inject statistics" into a computation of the autocorrelation function, and then we will know the power spectrum of our statistical signal from (32.9). All we have to do is Fourier transform the autocorrelation function $r_x(t)$.

(d) Verification of Wiener-Khintchine for a Square Pulse

Equation (32.3) gives the autocorrelation function $r_x(t)$ for a square pulse. One can insert this into the Fourier transform (1.1) to compute $R_x(\omega)$,

$$R_x(\omega) = \int_{-\tau}^{\tau} dt \, (A^2\tau) \, [\, 1 - |\, t \,| \,/\, \tau \,] \, e^{-i\omega t} \;\; = 2(A^2\tau) \int_{0}^{\tau} dt \, (1-t/\tau) \cos(\omega t)$$

$$= 2(A^2\tau)\,(1-\cos(\omega\tau))/(\omega^2\tau) = 4(A^2\tau)\,\sin^2(\omega\tau/2)/(\omega^2\tau) \;=\; (A^2\tau^2)\,\sin^2(\omega\tau/2)/(\omega\tau/2)^2$$

$$= (A\tau)^2 \, [\, \text{sinc}(\omega\tau/2)\,]^2, \qquad (32.11)$$

and this is recognized from (9.2) to be $|X(\omega)|^2$ for the square pulse, in agreement with (32.8) .

(e) Cross-correlation, convolution, and autocorrelation

Notation: a* means complex conjugation, $b \ast c$ means convolution, $b \star c$ means cross-correlation .

The **cross-correlation** of two functions b and c is defined this way

$$(b\star c)(t) \equiv \int_{-\infty}^{\infty} dt'\ b^*(t')c(t+t') = (c\star b)^*(-t) \tag{32.12}$$

where the right equality is easy to show setting t+t' = t". So in general, $b\star c \neq c\star b$. According to our **autocorrelation** definition (32.1),

$$r_{\mathbf{x}}(t) \equiv \int_{-\infty}^{\infty} dt'\ x(t')^*\ x(t'+t)\ , \tag{32.1}$$

the autocorrelation of function b is the cross-correlation of b with itself, so $r_{\mathbf{b}} = b\star b$.

In Section 7 we noted that a function is **Hermitian** if $f^*(-t) = f(t)$.

Fact: If b and c are both Hermitian, then $b\star c = c\star b$. (32.13)

Proof: $b\star c = \int_{-\infty}^{\infty} dt'\ b^*(t')c(t+t') = \int_{-\infty}^{\infty} dt'\ b(-t')c^*(-t-t') = \int_{-\infty}^{\infty} dt'\ b(t''+t)c^*(t'') = c\star b$

If b and c are both real and both even, they are both Hermitian so again, $b\star c = c\star b$.

The **convolution** of two functions b and c we saw from the (3.1) and (3.2) was this

$$(b\ast c)(t) = \int_{-\infty}^{\infty} dt'\ b(t')c(t-t') = (c\ast b)(t)\ .$$

In order to relate these two operations, we need to show more detail in the notation. Thus, using t" = -t',

$$[b(t)\star c(t)](t) = \int_{-\infty}^{\infty} dt'\ b^*(t')c(t+t') = \int_{-\infty}^{\infty} dt''\ b^*(-t'')c(t-t'') = [b^*(-t) \ast c(t)](t)\ .$$

If b(t) is a Hermitian function so $b^*(-t) = b(t)$, then we have shown that :

Fact: If b is Hermitian, then $b\star c = b\ast c$. (32.14)

If b = c = real, then we have from (32.1) and (32.2),

$$r_b(t) = \int_{-\infty}^{\infty} dt'\, b(t')b(t'+t) = \int_{-\infty}^{\infty} dt'\, b(t')b(t'-t)$$

so we have just proven part (a) of this fact, while part (b) was shown above (32.13),

Fact: (a) If b is real, then $r_b = b \star b = b * b$

(b) For any b, $r_b = b \star b$ (32.15)

(f) Z Transform Wiener-Khintchine relation for a Pulse Train

The Wiener-Khintchine relation was stated above in section (c) as

$$R_x(\omega) = |X(\omega)|^2 \qquad \textit{(32.8)}$$

where $R_x(\omega)$ is the Fourier Integral transform of the autocorrelation function $r_x(t)$

$$r_x(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')^* x(t'+t) \; . \qquad \textit{(32.1)}$$

in the case that x(t) is real. Once we know about the convolution theorem (3.6) for the Fourier Integral Transform, we see that the Wiener-Khintchine relation is just the application of this theorem to the particular convolution equation (32.1).

From the amplitudes y_n of an infinite pulse train one can define an **autocorrelation sequence** in analogy with the autocorrelation function,

$$r_s \equiv \lim_{N\to\infty} \left[\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n^* y_{n+s}\right] \equiv \langle y_n^* y_{n+s}\rangle_1 \quad . \qquad (32.16)$$

Although $r_x(t)$ (our particular (32.1) definition) has no normalization factor, we have added $\frac{1}{(2N+1)}$ to the definition of r_s in order to obtain the finite result $r_s = \langle y_n^* y_{n+s}\rangle_1$. The label 1 on $\langle ...\rangle_1$ reminds us that this is an average over a single sequence y_n . Later we shall use $\langle ...\rangle$ without a label to indicate an average over an ensemble.

It is convenient to use this shorthand notation for r_s,

$$r_s = \frac{1}{(2N+1)} \sum_{n=-\infty}^{\infty} y_n^* y_{n+s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n^* y_{n+s} \qquad (32.17)$$

where $T = (2N+1)T_1$ is the duration of the pulse train. We know that this infinite T is going to cancel another T in any "application" so we allow it to exist temporarily, as in (33.22) where $T = [2\pi\delta(0)]T_1$.

The Z transform of r_s is given by

$$R''(z) \equiv \sum_{s=-\infty}^{\infty} r_s z^{-s} = \sum_{s=-\infty}^{\infty} \{ \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n^* y_{n+s} \} z^{-s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} [y_n^* z^n] [y_{n+s} z^{-(n+s)}]$$

$$= \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [y_n^* z^n] [y_m z^{-m}] \qquad m \equiv n+s$$

$$= \frac{T_1}{T} [\sum_{n=-\infty}^{\infty} y_n^* z^n] [\sum_{m=-\infty}^{\infty} y_m z^{-m}] = \frac{T_1}{T} Y''(z)^* Y''(z) = \frac{T_1}{T} | Y''(z) |^2$$

so we have obtained a Z Transform version of the Wiener-Khintchine relation. We may regard this simple result as an application of the Z Transform convolution theorem (24.5) to the particular convolution sum (32.17) with $\Delta t \rightarrow \frac{T_1}{T}$. Here is comparison of the two cases:

$$R''(z) = \frac{T_1}{T} | Y''(z) |^2 \qquad \text{Z Transform Wiener-Khintchine} \qquad (32.18)$$

$$R_x(\omega) = |X(\omega)|^2 \ . \qquad \text{regular Wiener-Khintchine} \qquad \textit{(32.8)}$$

Here $X(\omega)$ is the Fourier Integral Transform of the pulse train $x(t)$,

$$x(t) = \sum_{n=-\infty}^{\infty} y_n x_{pulse}(t - t_n). \qquad \textit{(25.1)}$$

while $Y''(z)$ is the Z transform of the sequence of pulse train amplitudes.

The results of this section apply to **finite** as well as infinite pulse trains. In a way, this fact is obvious if we just pad out the finite pulse train with 0's to make it infinite, but it is still worth showing explicitly. The appropriate autocorrelation sequence for a finite pulse train is given by,

$$r_s \equiv \frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n^* y_{n+s} \equiv \langle y_n^* y_{n+s} \rangle_1 \qquad \text{.// finite pulse train} \qquad (32.16)'$$

If we assume y_n vanishes outside the range -N to N, then r_s vanishes outside the range -2N to 2N, in accord with the analog comment above that $r_x(t)$ has twice the width of any finite $x(t)$. We shall nevertheless show Σ_s below as having an infinite range, though we know the range will really be finite when the summand is examined.

We then compute $R''(z)$ as above

$$R''(z) \equiv \sum_{s=-\infty}^{\infty} r_s\, z^{-s} = \sum_{s=-\infty}^{\infty} \{ \frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n^* \, y_{n+s} z^{-s} \} = \frac{1}{(2N+1)} \sum_{s=-\infty}^{\infty} \sum_{n=-N}^{N} y_n^* \, y_{n+s}\, z^{-s}$$

$$= \frac{1}{(2N+1)} \sum_{n=-N}^{N} \sum_{s=-\infty}^{\infty} y_n^* \, y_{n+s}\, z^{-s}$$

Now let m = n+s and replace the s sum with an m sum,

$$= \frac{1}{(2N+1)} \sum_{n=-N}^{N} \sum_{m=\infty}^{-\infty} y_n^* \, y_m\, z^{-(m-n)}$$

But now the range restrictions from y_m on m gives us this final result

$$= \frac{1}{(2N+1)} \sum_{n=-N}^{N} \sum_{m=-N}^{N} y_n^* \, y_m\, z^{-(m-n)} = \frac{1}{(2N+1)} (\sum_{n=-N}^{N} y_n^* \, z^{n}) \; (\sum_{m=-N}^{N} y_m\, z^{-m})$$

so then

$$R''(z) = \frac{1}{(2N+1)} \mid Y''(z) \mid^2 = \frac{T_1}{T} \mid Y''(z) \mid^2 \qquad \text{where now } T = (2N+1)T_1. \qquad (32.18)'$$

which has the exact same form as (32.18).

33. Spectral Power Density of a Simple Pulse Train

In subsections (a) through (d) we deal only with simple pulse trains. Along the way, certain facts are developed which apply to general as well as simple pulse trains and these will be gathered up and summarized in the first part of Section 34.

Comment: Some texts use the phrase "power spectral density" (PSD). Although this wins on Google by a ratio of 3 to 1, we still prefer the phrase "spectral power density", and the same for "spectral energy density".

(a) Computation of $X(\omega)$ and $|X(\omega)|^2$ for an *infinite* Simple Pulse Train

This is the first section in which we use the $\delta(0)$ notation of Appendix A which may make the reader feel a bit uncomfortable. In subsection (b) we shall repeat everything for a finite pulse train and then take the limit $N\to\infty$ to obtain the same results without using $\delta(0)$. In both sections we shall include the Z Transform in passing, but our main work is in the ω variable, not the z variable.

From Section 14 (a) we know the spectrum of an infinite pulse train formed from pulses $x_{pulse}(t)$ separated by time T_1 ,

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) \qquad \textit{(14.1)}$$

$$X(\omega) = X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\ \delta(\omega T_1 - 2\pi m) \ . \qquad \textit{(14.4)}$$

We can obtain the same expressions from box (25.4) which summarizes amplitude modulated pulse trains by setting all amplitudes to $y_n = 1$,

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\ x_{pulse}(t - t_n) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - t_n) \qquad (33.1)$$

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\ Y'(\omega) = X_{pulse}(\omega)\ Y''(z) \qquad (33.2)$$

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega nT_1} = \sum_{n=-\infty}^{\infty} e^{-i\omega nT_1} \ . \qquad (33.3)$$

$Y'(\omega)$ is the Digital Fourier Transform of $y_n = 1$, and $Y''(z)$ is the Z Transform, where $z = e^{i\omega T_1}$.

In (33.3) we then use (13.2)

$$\sum_{n=-\infty}^{\infty} e^{\pm ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad \textit{(13.2)}$$

so that (33.3) becomes

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad (33.4)$$

and then (33.2) says

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\, Y'(\omega) = X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad (33.5)$$

in agreement with (14.4) quoted just above (33.1).

To find the power spectrum of a signal x(t), our first task is to compute $| X(\omega) |^2$. From (33.2) we get

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2 (1/T_1)^2 |Y'(\omega)|^2 = |X_{pulse}(\omega)|^2 | Y''(z) |^2 . \qquad z = e^{i\omega T_1} \qquad (33.6)$$

We therefore must deal with the following object, using (33.4),

$$| Y''(z) |^2 = (1/T_1)^2 |Y'(\omega)|^2 = [\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)]^2 , \qquad (33.7)$$

and we are now faced with the issue of squaring delta functions. Formally these objects don't exist in the realm of distribution theory, but (as discussed in Appendix A) we can deal with them in an *ad hoc* way which proves to be useful. Consider,

$$[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)]^2 = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(\omega T_1 - 2\pi n) .$$

Looking at the product of the two delta functions, there can be no contribution to the double sum unless m = n, so we continue

$$= \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\, 2\pi\delta(0) = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

so that

$$[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)]^2 = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) . \qquad (33.8)$$

The object $\delta(0)$ is formally undefined, but in Appendix A we ascribe the meaning that $2\pi\delta(0) = 2N+1$ in the limit that $N \to \infty$ and we can always "undo the limit" when necessary. We shall firm up this idea in section (b) directly below. So we have shown then that

$$| Y''(z) |^2 = (1/T_1)^2 |Y'(\omega)|^2 = [2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \tag{33.9}$$

or

$$\frac{|Y''(z)|^2}{[2\pi\delta(0)]} = (1/T_1)^2 \frac{|Y'(\omega)|^2}{[2\pi\delta(0)]} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \ . \tag{33.10}$$

Then from (33.6)

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 \frac{|Y''(z)|^2}{[2\pi\delta(0)]} = |X_{pulse}(\omega)|^2 (1/T_1)^2 \frac{|Y'(\omega)|^2}{[2\pi\delta(0)]}$$

$$= |X_{pulse}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \ . \tag{33.11}$$

We shall now repeat the above set of steps for a finite pulse train.

(b) Computation of X(ω) and $|X(\omega)|^2$ for a *finite* Simple Pulse Train

Our finite pulse train always has pulses ranging from n = -N to N instead of from n = -∞ to ∞. We start off exactly as in the previous section but with limited sums,

$$x(t) = \sum_{n=-N}^{N} y_n\ x_{pulse}(t - t_n) = \sum_{n=-N}^{N} x_{pulse}(t - t_n) \tag{33.12}$$

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\ Y'(\omega) = X_{pulse}(\omega)\ Y''(z) \tag{33.13}$$

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-N}^{N} y_n\, e^{-i\omega n T_1} = \sum_{n=-N}^{N} e^{-i\omega n T_1} \ . \tag{33.14}$$

In the last line we then use (13.3),

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \qquad -\infty < k < \infty \tag{13.3}$$

where δ_5 is a periodic delta function model discussed in Appendix A (b). Equation (33.14) becomes

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi\, \delta_5(\omega T_1, N) \tag{33.15}$$

and then equation (33.13) says

$$X(\omega) = X_{pulse}(\omega)\, 2\pi\, \delta_5(\omega T_1,N) . \tag{33.16}$$

This δ_5 is periodic with period 2π and has identical peaks separated by 2π. For finite N, these are peaks of finite width and height. Squaring, we find

$$|\, X(\omega)\, |^2 = |\, X_{pulse}(\omega)\, |^2\ [2\pi\, \delta_5(\omega T_1,N)]^2 \quad . \tag{33.17}$$

Recalling the definition of the δ_6 delta function model from Appendix A (A.20),

$$2\pi\, \delta_6(k,N) \equiv \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} \tag{A.20}$$

we obtain

$$\frac{|X(\omega)|^2}{(2N+1)} = |\, X_{pulse}(\omega)\, |^2\ \ 2\pi\, \delta_6(\omega T_1,N) \tag{33.18}$$

and this is the finite pulse train result. We can then take the limit $N\to\infty$ and make use of (A.21),

$$\lim_{N\to\infty} \delta_6(\omega T_1,N) = \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m) \tag{A.21}$$

to find that

$$\lim_{N\to\infty} \left[\frac{|X(\omega)|^2}{(2N+1)}\right] = |\, X_{pulse}(\omega)\, |^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \tag{33.19}$$

and this replicates (33.11) with the promised connection $2\pi\delta(0) = \lim_{N\to\infty} (2N+1)$.

It is useful now to provide some side-by-side comparisons of results :

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} e^{-i\omega n T_1} = \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \tag{33.4}$$

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-N}^{N} e^{-i\omega n T_1} = 2\pi\, \delta_5(\omega T_1,N) \qquad \text{finite} \tag{33.15}$$

$$X(\omega) = X_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \tag{33.5}$$

$$X(\omega) = X_{pulse}(\omega)\ \ 2\pi\, \delta_5(\omega T_1,N) . \qquad \text{finite} \tag{33.16}$$

$$\frac{|X(\omega)|^2}{[2\pi\delta(0)]} = |X_{\mathtt{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \qquad \text{infinite} \qquad (33.11)$$

$$\frac{|X(\omega)|^2}{(2N+1)} = |\, X_{\mathtt{pulse}}(\omega)\,|^2 \;\; 2\pi\, \delta_6(\omega T_1,N) \qquad \text{finite} \qquad (33.18)$$

One can interpret $\frac{|X(\omega)|^2}{[2\pi\delta(0)]}$ as the value of $|X(\omega)|^2$ *per pulse* in an infinite pulse train.

(c) Spectral Power Density of a Simple Pulse Train

In this section, everything is in the frequency domain, nothing is in the time domain.

Recall from (32.6) that

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = \text{energy density in the } \omega\text{-domain} \qquad \text{(joule-sec)} \qquad (32.6)$$

$$\mathcal{E}(\omega)d\omega = \text{energy in } d\omega \qquad \text{(joules)}$$

where $\mathcal{E}(\omega)$ is the spectral energy density of signal x(t) whose Fourier Transform is X(ω).

If we divide $\mathcal{E}(\omega)$ by 2N+1 or $2\pi\delta(0)$ we obtain the pulse train's average spectral energy density per pulse, which is the same as the energy density of an average pulse in the pulse train. Using our two expressions above for infinite and finite pulse trains, we then find

$$\frac{\mathcal{E}(\omega)}{[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite}$$

$$\frac{\mathcal{E}(\omega)}{(2N+1)} = \frac{|X(\omega)|^2}{2\pi(2N+1)} = \frac{1}{2\pi}\,|\, X_{\mathtt{pulse}}(\omega)\,|^2 \;\; 2\pi\, \delta_6(\omega T_1,N) \quad . \qquad \text{finite} \qquad (33.20)$$

If we divide the spectral energy density of the average pulse by T_1, we obtain the spectral power density of an average pulse, and this is the same as the average spectral power density of the pulse train. Thus,

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_1[2\pi\delta(0)]} = \frac{|X(\omega)|^2}{2\pi T_1[2\pi\delta(0)]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \,(1/T_1) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T_1[2N+1]} = \frac{|X(\omega)|^2}{2\pi T_1[2N+1]} = \frac{1}{2\pi}|X_{\mathtt{pulse}}(\omega)|^2 \,(1/T_1)\;\; 2\pi\, \delta_6(\omega T_1,N) \quad . \qquad \text{finite}$$

(33.21)

If is perhaps helpful to define

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \quad . \qquad (33.22)$$

Then (33.21) maybe be restated

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{pulse}(\omega)|^2\,(1/T_1)\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{infinite}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = \frac{1}{2\pi}|X_{pulse}(\omega)|^2\,(1/T_1)\;2\pi\,\delta_6(\omega T_1,N) \quad . \qquad \text{finite} \qquad (33.23)$$

Meanwhile, our $x_{pulse}(t)$ which lasts only for duration T_1 itself has an energy and power density,

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \quad . \qquad (33.24)$$

Note: Even if $x_{pulse}(t)$ is wider than T_1 as the Gaussians are in Fig 14.2, the pulse is *associated with* the interval of width T_1, and $X_{pulse}(\omega)$ involves the time integral over all of $x_{pulse}(t)$. It is convenient to think of the pulse as the black curve in Fig 14.2, in which case the pulse really fits within T_1.

We can then write (33.23) in the following compact form

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules} \qquad \text{infinite}$$

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\;2\pi\,\delta_6(\omega T_1,N) \qquad \text{joules} \qquad \text{finite} \qquad (33.25)$$

Notice that $\delta(\omega T_1\text{-}2\pi m)$ and $\delta_6(\omega T_1,N)$ are both dimensionless, so the dimensions in each equation trivially match. A power density $\mathcal{P}(\omega)$ has dimensions of energy = joules, so that $\mathcal{P}(\omega)d\omega$ then has the dimensions of joules/sec = watts, and this is the pulse train power contained in interval $d\omega$ of the spectrum.

For the infinite pulse train, we are always allowed to write

$$2\pi\,\delta(\omega T_1 - 2\pi m) = (2\pi/T_1)\,\delta(\omega - m(2\pi/T_1)) = \omega_1\,\delta(\omega - m\omega_1) \qquad \omega_1 \equiv 2\pi/T_1$$

to get

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\,\omega_1\sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_1) \qquad \text{infinite} \qquad (33.26)$$

which shows more explicitly that the power lines occur at the harmonics $\omega = m\omega_1$. Recall now these earlier facts,

$$c(\omega) \equiv (1/T_1)X_{pulse}(\omega) \qquad \textit{(14.14)}$$

$$c_m \equiv c(m\omega_1) = (1/T_1)X_{pulse}(m\omega_1) \ . \qquad \textit{(14.10), (14.8)}$$

For the infinite pulse train we can then write, using (33.24),

$$\mathcal{P}(\omega) \equiv \mathcal{P}_{pulse}(\omega)\, \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \frac{2\pi}{T_1} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)$$

$$= \frac{|X_{pulse}(\omega)|^2}{T_1^2} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = |c(\omega)|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) = \sum_{m=-\infty}^{\infty} |c(m\omega_1)|^2\, \delta(\omega - m\omega_1)$$

so that

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\, \delta(\omega - m\omega_1). \qquad (33.27)$$

This gives the power spectrum of a simple pulse train in terms of the complex Fourier Series coefficients c_m.

Recall from box (15.12) that $\mathrm{Dim}(c_m) = \mathrm{Dim}[x(t)]$ = volts (say), so $|c_m|^2$ = watts into a 1Ω resistor, and since $\delta(\omega - m\omega_1)$ has dimensions sec, $|c_m|^2\, \delta(\omega - m\omega_1)$ then has dimensions watt-sec = joules, as befits any $\mathcal{P}(\omega)$ object.

If we assume x(t) is a real pulse train, then X(ω) is Hermitian, $X(-\omega) = [\,X(\omega)]^*$ by (7.3), which means

$$c_{-m} = (1/T_1)X_{pulse}(-m\omega) = (1/T_1)[X_{pulse}(m\omega)]^* = c_m^*$$

so

$$|c_{-m}|^2 = |c_m|^2 \qquad x(t) \text{ real} \qquad (33.28)$$

and then we can fold the negative part of the sum in (33.27) to get

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\, \delta(\omega - m\omega_1) = |c_0|^2\, \delta(\omega) + 2 \sum_{m=1}^{\infty} |c_m|^2\, \delta(\omega - m\omega_1)\ . \qquad (33.29)$$

Finally, recalling from our Fourier Series box (15.2) that $c_m = [\, a_m - ib_m\,]/2$ and $b_0 = 0$, we get

$$\mathcal{P}(\omega) = a_0^2\, \delta(\omega) + (1/2) \sum_{m=1}^{\infty} (a_m^2 + b_m^2)\, \delta(\omega - m\omega_1)\ . \qquad (33.30)$$

(d) Average Power P of a Simple Pulse Train

We seek an expression for the average power P in a general pulse train. This is of course a time domain quantity, not a frequency domain quantity.

P = [total energy in pulse train / time duration of pulse train] = average pulse train power

$$= (1/T) \int_{-\infty}^{\infty} dt\, |x(t)|^2 = \int_{-\infty}^{\infty} d\omega\, \frac{|X(\omega)|^2}{2\pi T} \qquad \text{// from (32.4) with } R = 1\Omega$$

so

$$P = \int_{-\infty}^{\infty} d\omega\, \mathscr{P}(\omega) \quad . \qquad \text{// from (33.23)} \qquad (33.31)$$

This is certainly reasonable since $\mathscr{P}(\omega)$ is the average spectral power density of the pulse train. Recall that $|x(t)|^2$ is the "instantaneous power" of signal x(t) at some instant of time t, so P is the time average of this instantaneous power.

For the special case of an infinite simple pulse train, we found above that

$$\mathscr{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\ \delta(\omega - m\omega_1) = a_0^2\ \delta(\omega) + (1/2) \sum_{m=1}^{\infty} (a_m^2+b_m^2)\ \delta(\omega - m\omega_1) \quad .$$

Therefore the power in a simple pulse train is given by (33.31) as

$$P = \sum_{m=-\infty}^{\infty} |c_m|^2 = a_0^2 + (1/2) \sum_{m=1}^{\infty} (a_m^2+b_m^2) \ . \qquad (33.32)$$

34. Spectral power density of a General Pulse Train

(a) General Pulse Train results and connection with the Autocorrelation Function

Certain results of the previous section apply to general pulse trains. They are gathered here:

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = \text{energy density in the } \omega\text{-domain} \qquad \text{(joule-sec)} \qquad \textit{(32.6)}$$

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad \textit{(33.22)}$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad \textit{(33.23)}$$

$$\mathcal{P}_{\mathbf{pulse}}(\omega) \equiv \frac{\mathcal{E}_{\mathbf{pulse}}(\omega)}{T_1} = \frac{|X_{\mathbf{pulse}}(\omega)|^2}{2\pi T_1} \qquad \textit{(33.24)}$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) \quad . \qquad \textit{(33.31)}$$

We now bring the autocorrelation function into the discussion, but only in passing. Let x(t) be an arbitrary but real pulse train, and recall that

$$r_x(t) \equiv \int_{-\infty}^{\infty} dt'\, x(t')\, x(t'+t) \quad . \qquad \textit{(32.1)}$$

Then, using T from (33.22),

$$r_x(0) \equiv \int_{-\infty}^{\infty} dt'\, x(t')^2 = P\,T = E = \text{total energy in the pulse train} \qquad (34.1)$$

so the average pulse train power maybe written in terms of the autocorrelation function evaluated at t = 0,

$$P = r_x(0)/T \quad . \qquad (34.2)$$

We diagonalized (32.1) treated as a convolution equation to obtain

$$|X(\omega)|^2 = R_x(\omega),$$

called the Wiener-Khintchine relation. Here $R_x(\omega)$ is the Fourier Integral Transform of the autocorrelation function $r_x(t)$ of x(t). Then from (33.23) that $\mathcal{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T}$ we get

$$\mathcal{P}(\omega) = R_x(\omega)/\,(2\pi T) \,. \qquad (34.3)$$

In this way, both P and $\mathcal{P}(\omega)$ can be expressed in terms of the autocorrelation function. Thus, one approach to finding P and $\mathcal{P}(\omega)$ for a pulse train is to try and determine $r_x(t)$.

Here then is a box summarizing all the general pulse train results:

Energy and Power Properties of a General Pulse Train (34.4)

$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi$ = energy density in the ω-domain (joule-sec) *(32.6)*

$$T \equiv \begin{cases} [2\pi\delta(0)]T_1 & \text{infinite pulse train} \\ (2N+1)T_1 & \text{finite pulse train} \end{cases} \qquad (33.22)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} = R_x(\omega)/(2\pi T) \quad \text{joules} \qquad (33.23) \text{ and } (34.3)$$

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \qquad (33.24)$$

$$P = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) = r_x(0)/T \quad \text{watts} \qquad (33.31) \text{ and } (34.2)$$

If $P(f)df = \mathcal{P}(\omega)d\omega = \mathcal{P}(\omega)\, 2\pi df$, then $P(f) = 2\pi\mathcal{P}(\omega)$.

(b) Spectral power density for a General Pulse Train

In Section 33 (d) we dealt with simple pulse trains. Here we consider the more general amplitude modulated pulse train. In all equations, one can replace $\sum_{n=-\infty}^{\infty}$ by $\sum_{n=-N}^{N}$ to adapt the equation to a finite pulse train instead of an infinite one. We start then with

$$x(t) = \sum_{n=-\infty}^{\infty} y_n\, x_{pulse}(t - t_n) \qquad (25.1) \qquad (34.5)$$

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\, Y'(\omega) = X_{pulse}(\omega)\, Y''(z) \qquad (25.3) \qquad (34.6)$$

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} . \qquad (24.2) \qquad (34.7)$$

To find the frequency-domain power spectrum of a signal x(t), our first task is to compute $|X(\omega)|^2$. From (34.6) we get

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2 (1/T_1)^2 |Y'(\omega)|^2 = |X_{pulse}(\omega)|^2 |Y''(z)|^2 \qquad z = e^{i\omega T_1} \qquad (34.8)$$

We therefore must deal with the following object, using (34.7),

$$| Y''(z) |^2 = (1/T_1)^2 |Y'(\omega)|^2 = | \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} |^2$$

$$= \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \sum_{m=-\infty}^{\infty} y_m^* e^{+i\omega m T_1} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.9)$$

so that

$$| X(\omega) |^2 = | X_{pulse}(\omega) |^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \quad . \qquad (34.10)$$

From box (34.4) we then have

$$\mathcal{E}(\omega) \equiv |X(\omega)|^2/2\pi = (1/2\pi) | X_{pulse}(\omega) |^2 \; | Y''(z) |^2$$

$$= (1/2\pi) | X_{pulse}(\omega) |^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.11)$$

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = (1/2\pi T) | X_{pulse}(\omega) |^2 | Y''(z) |^2$$

$$= (1/2\pi T) | X_{pulse}(\omega) |^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.12)$$

which we can write as (again using box (34.4) results)

$$\mathcal{E}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) \; | Y''(z) |^2 = T_1 \mathcal{P}_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.13)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \; | Y''(z) |^2 = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.14)$$

$$P = \int_{-\infty}^{\infty} d\omega \, \mathcal{P}(\omega) \quad . \qquad \textit{(33.31)} \qquad (34.15)$$

Using the Z Transform Wiener-Khintchine relation (32.18) that $R''(z) = \frac{T_1}{T} | Y''(z) |^2$ we get

$$\mathcal{E}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) \; | Y''(z) |^2 = T \mathcal{P}_{pulse}(\omega) R''(z) \qquad (34.13a)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \; | Y''(z) |^2 = \mathcal{P}_{pulse}(\omega) R''(z) \qquad (34.14a)$$

where R"(z) is the Z transform of the autocorrelation sequence r_s obtained from the y_n.

Not knowing details of the y_n there is not much else we can do in these expressions.

(c) Pulse Trains with Repeated Sequences

Consider a pulse train composed of some general pulse shape $x_{pulse}(t)$ whose amplitudes are repeated sequences of A,B. We shall compute the spectrum $X(\omega)$ and spectral power density $\mathcal{P}(\omega)$ by two different methods.

The first method is more or less brute force, and it reveals a potential pitfall in using the $\delta(0)$ notation and shows a clean way to avoid the pitfall.

The second method, much simpler, is to use the Fourier Series results in box (15.12) applied to the repeating sequence.

We then state $X(\omega)$ and $\mathcal{P}(\omega)$ for a few special cases including various square waves.

Appendix F treats the general case of a repeated sequence {A,B,C,D......}.

Method 1: Brute Force Calculation of $X(\omega)$ and $\mathcal{P}(\omega)$ for repeated A,B case.

Our starting point is (34.8) with (34.7), where we assume N is large and later we will take $N\to\infty$:

$$X(\omega) = (1/T_1)X_{pulse}(\omega)\, Y'(\omega) = X_{pulse}(\omega)\, Y''(z) \tag{34.6}$$

$$|X(\omega)|^2 = |X_{pulse}(\omega)|^2 (1/T_1)^2 |Y'(\omega)|^2 = |X_{pulse}(\omega)|^2 \,| Y''(z) |^2 \qquad z = e^{i\omega T_1} \tag{34.8}$$

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-N}^{N} y_n e^{-i\omega n T_1}. \tag{34.7}$$

The main problem is to compute $Y''(z)$ and then square it. We have

$$\sum_{n=-N}^{N} y_n e^{-i\omega n T_1} = A \sum_{n\ \mathrm{even}} e^{-i\omega n T_1} + B \sum_{n\ \mathrm{odd}} e^{-i\omega n T_1} .$$

Now process the sums as follows, where

$$\sum_{n\ \mathrm{even}} e^{-i\omega n T_1} = \sum_{m=-N/2}^{N/2} e^{-i\omega (2m) T_1} \quad \text{where we used } n = 2m$$

$$\sum_{n\ \mathrm{odd}} e^{-i\omega n T_1} = \sum_{n=(-N-1)/2}^{(N-1)/2} e^{-i\omega (2m+1) T_1} \quad \text{where we used } n = 2m + 1 \ .$$

We assume N is very large, so we regard $(N\pm1)/2 \approx N/2$. We then find

$$Y''(z) = Y'(\omega)/T_1 = \sum_{n=-N}^{N} y_n e^{-i\omega n T_1} = [\, A + B\, e^{-i\omega T_1}] \sum_{m=-N/2}^{N/2} e^{-i\omega (2m) T_1} .$$

We now use (13.3),

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \ , \qquad -\infty < k < \infty \qquad (13.3)$$

to write

$$\sum_{m=-N/2}^{N/2} e^{-i\omega(2m)T_1} = 2\pi\, \delta_5(2\omega T_1, N/2)$$

where δ_5 (and δ_6 to come soon) are explained in Appendix A (b). Therefore,

$$Y''(z) = [\, A + B\, e^{-i\omega T_1}]\, 2\pi\delta_5(2\omega T_1, N/2) \quad . \qquad (34.16)$$

Using this Appendix A result,

$$\lim_{N\to\infty} \delta_5(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.19)$$

we obtain the N→∞ limit for our spectrum

$$\begin{aligned} Y''(z) &= [\, A + B\, e^{-i\omega T_1}]\, 2\pi \sum_{m=-\infty}^{\infty} \delta(2\omega T_1 - 2\pi m) \\ &= (1/2)[\, A + B\, e^{-i\omega T_1}]\, (1/T_1)\ 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \\ &= (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} [\, A + B\, (-1)^m]\ \delta(\omega - m\omega_1/2) \qquad (34.17) \end{aligned}$$

and correspondingly

$$X(\omega) = X_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} [\, A + B(-1)^m\,]\, \delta(\omega - m\omega_1/2)\ . \qquad (34.18)$$

There is a certain logic to the $[\, A + B\, e^{-i\omega T_1}]$ factor in (34.17). If we set A = K and B = 0 we get one result, and if we set A = 0 and B = K we get the same result multiplied by $e^{-i\omega n T_1}$. The second pulse train is just the first pulse train shifted T_1 units to the right, and this adds phase $e^{-i\omega n T_1}$ as in (12.1).

If we were to square (34.18) and use our usual $2\pi\delta(0) = 2N+1$ association, we get a result that is off by a factor of 2. The reason is that our pre-limit sums are going from -N/2 to N/2, so we would get the right answer if we were to adjust and say $2\pi\delta(0) = N+1$. Rather than make an arm-waving argument to this effect, it is safer to continue along with our pre-limit expressions, having paused to take the limit for the spectrum X(ω) as in (34.18).

So, backing off again from limit, we square (34.16) to get

$$|Y''(z)|^2 = |A + Be^{-i\omega T_1}|^2\, [2\pi\, \delta_5(2\omega T_1,N/2)]^2 \,.$$

Then from (A.20) applied with N → N/2

$$\delta_6(k,N/2) \equiv \frac{1}{2\pi}\ \frac{[2\pi\delta_5(k,N/2)]^2}{(N+1)} \qquad (A.20)$$

we get

$$|Y''(z)|^2 = |A + Be^{-i\omega T_1}|^2\, [2\pi\, \delta_5(2\omega T_1,N/2)]^2$$

or

$$\frac{|Y''(z)|^2}{2\pi(N+1)} = |A + Be^{-i\omega T_1}|^2 \left\{\frac{1}{2\pi}\ \frac{[2\pi\delta_5(2\omega T_1,N/2)]^2}{(N+1)}\right\} = |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1,N/2)\,.$$

Now for large N we ignore the difference between N and N + 1 and so on, so we divide both sides by 2 to get,

$$\frac{|Y''(z)|^2}{2\pi(2N+1)} = (1/2)\, |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1,N/2)\ .$$

Notice that a very important factor of 1/2 appears on the right in the last step. We now insert the squared pulse spectrum to get

$$\frac{|X(\omega)|^2}{2\pi(2N+1)} = |X_{pulse}(\omega)|^2\, \frac{|Y''(z)|^2}{2\pi(2N+1)} = |X_{pulse}(\omega)|^2\ (1/2)\, |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1,N/2) \quad .$$

If we divide both sides by T_1 the left side is $\frac{|X(\omega)|^2}{2\pi T}$ where T is the length of the pulse train and this in turn equals $\mathcal{P}(\omega)$, all as shown in box (34.4). So for large N we have shown that

$$\mathcal{P}(\omega) = |X_{pulse}(\omega)|^2\, (1/T_1)(1/2)\, |A + Be^{-i\omega T_1}|^2\ \delta_6(2\omega T_1,N/2)$$

$$= \mathcal{P}_{pulse}(\omega)(1/2)\, |A + Be^{-i\omega T_1}|^2\ 2\pi\, \delta_6(2\omega T_1,N/2)\ . \qquad (34.19)$$

Now at last we take the limit N→∞ and use

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.21)$$

to get our desired infinite pulse train result

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)(1/2)\,|A + Be^{-i\omega T_1}|^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(2\omega T_1 - 2\pi m)$$

$$= \mathscr{P}_{pulse}(\omega)(1/4)\,|A + Be^{-i\omega T_1}|^2\,(1/T_1) \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega - m\omega_1/2)$$

$$= \mathscr{P}_{pulse}(\omega)(1/4)\,(1/T_1) \sum_{m=-\infty}^{\infty} |A + B(-1)^m|^2\; 2\pi\,\delta(\omega - m\omega_1/2)$$

$$= \mathscr{P}_{pulse}(\omega)(1/4)\,\omega_1 \sum_{m=-\infty}^{\infty} \{\, |A|^2 + |B|^2 + 2\mathrm{Re}(AB^*)(-1)^m \,\}\; \delta(\omega - m\omega_1/2)\,. \qquad (34.20)$$

Summarizing the key results:

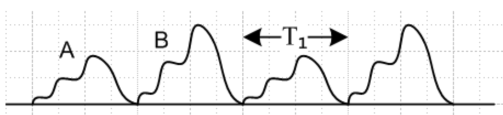


Fig 34.1

$$X(\omega) = X_{pulse}(\omega)\,(1/2)\,\omega_1 \sum_{m=-\infty}^{\infty} [\, A + B(-1)^m \,]\;\delta(\omega - m\omega_1/2)\,. \qquad \textit{(34.18)}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\,(1/4)\,\omega_1 \sum_{m=-\infty}^{\infty} \{\, |A|^2 + |B|^2 + 2\mathrm{Re}(AB^*)(-1)^m \,\}\;\delta(\omega - m\omega_1/2)\,. \qquad \textit{(34.20)}$$

Method 2: Fourier Series Calculation of $X(\omega)$ and $\mathscr{P}(\omega)$ for repeated A,B case.

We regard our A,B pair of pulses as a single pulse $x_{PULSE}(t)$ of width $2T_1$.

$$x_{PULSE}(t) = \begin{cases} A\,x_{pulse}(t) & 0 < t < T_1 \\ B\,x_{pulse}(t-T_1) & T_1 < t < 2T_1 \end{cases}$$

We then apply the Fourier Series results of box (15.12) but with $T_1 \rightarrow 2T_1$ (so $\omega_1 \rightarrow \omega_1/2$)

$$x(t) = \sum_{n=-\infty}^{\infty} x_{PULSE}(t - n2T_1)$$

$$C_m = (1/2T_1) \int_0^{2T_1} dt\; x_{PULSE}(t)\, e^{-im\omega_1 t/2}\,.$$

We then calculate C_m as follows

$$C_m = (1/2T_1) \int_0^{2T_1} dt\, x_{PULSE}(t)\, e^{-im\omega_1 t/2}$$

$$= (1/2T_1)\, A \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2} + (1/2T_1)\, B \int_{T_1}^{2T_1} dt\, x_{pulse}(t-T_1)\, e^{-im\omega_1 t/2}$$

$$= (1/2T_1)\, A \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2} + (1/2T_1)\, Be^{-im\omega_1 (T_1/2)} \int_0^{T_1} dt'\, x_{pulse}(t')\, e^{-im\omega_1 t'/2}$$

$$= (1/2)\ [\, A + B\, (-1)^m\,]\ (1/T_1) \int_0^{T_1} dt\, x_{pulse}(t)\, e^{-im\omega_1 t/2}$$

$$= (1/2)\ [\, A + B\, (-1)^m\,]\, c_{m/2}$$

so that

$$C_m = (1/2)\ [\, A + B\, (-1)^m\,]\, c_{m/2}$$

where c_n are the Fourier coefficients for a simple pulse train made from $x_{pulse}(t)$ pulses. The spectrum can then be read from box (14.12) item 3, where we continue to replace $T_1 \rightarrow 2T_1$,

$$X(\omega) = \sum_{m=-\infty}^{\infty} C_m\ 2\pi\ \delta(\omega - m\omega_1/2)\ .$$

But from (14.10) and (14.8) we know that

$$c_{m/2} = (1/T_1) X_{pulse}(m\omega_1/2)$$

so that

$$C_m = (1/2)\ [\, A + B\, (-1)^m\,]\ (1/T_1) X_{pulse}(m\omega_1/2)\ .$$

Then

$$X(\omega) = \sum_{m=-\infty}^{\infty} \{\, (1/2)\ [\, A + B\, (-1)^m\,]\ (1/T_1) X_{pulse}(m\omega_1/2)\}\ \ 2\pi\ \delta(\omega - m\omega_1/2)$$

$$= X_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} [\, A + B\, (-1)^m\,]\, \delta(\omega - m\omega_1/2)$$

which agrees with our Method 1 result (34.18).

Then from (33.29) we get

$$\begin{aligned}\mathcal{P}(\omega) &= \sum_{m=-\infty}^{\infty} |C_m|^2\ \delta(\omega - m\omega_1/2) \\ &= \sum_{m=-\infty}^{\infty} |\ (1/T_1)X_{pulse}(m\omega_1/2)\ \ (1/2)\ \ [\ A + B(-1)^m\ ]\ |^2\ \ \delta(\omega - m\omega_1/2) \\ &= (1/T_1)^2\ |\ X_{pulse}(\omega)\ |^2\ (1/4) \sum_{m=-\infty}^{\infty} |\ [\ A + B(-1)^m\ ]\ |^2\ \delta(\omega - m\omega_1/2) \quad \text{// now use (33.24)} \\ &= \mathcal{P}_{pulse}(\omega)\ (1/4)\ \omega_1 \sum_{m=-\infty}^{\infty} \{\ |A|^2 + |B|^2 + 2\mathrm{Re}(AB^*)(-1)^m\ \}\ \delta(\omega - m\omega_1/2)\end{aligned}$$

and this agrees with our Method 1 result (34.20).

Special Case 1 : Symmetric and Square Waves

Suppose A = 1 and B = -1. Then

$$\{ |A|^2 + |B|^2 + 2\mathrm{Re}(AB^*)(-1)^m \} = \{ 1 + 1 - 2(-1)^m \} = 2 [1 - (-1)^m] .$$

Our general results were these

$$X(\omega) = X_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} [A + B(-1)^m]\, \delta(\omega - m\omega_1/2) . \tag{34.18}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, (1/4)\, \omega_1 \sum_{m=-\infty}^{\infty} \{ |A|^2 + |B|^2 + 2\mathrm{Re}(AB^*)(-1)^m \}\, \delta(\omega - m\omega_1/2) \tag{34.20}$$

which become

$$X(\omega) = X_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} [1 - (-1)^m]\, \delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} \{ 1 - (-1)^m \}\, \delta(\omega - m\omega_1/2)$$

or

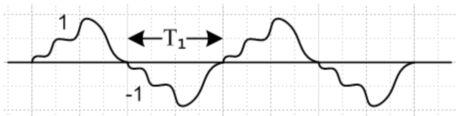


Fig 34.2

$$X(\omega) = X_{pulse}(\omega)\, \omega_1 \sum_{m=\pm odd} \delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \omega_1 \sum_{m=\pm odd} \delta(\omega - m\omega_1/2) . \tag{34.21}$$

<u>Square wave pulse train with peak-to-peak = 2 units and period $2T_1$</u>

$$X_{pulse}(\omega) = T_1\, \mathrm{sinc}(\omega T_1/2) \tag{9.2}$$

$$\mathcal{P}_{pulse}(\omega) = |X_{pulse}(\omega)|^2/ (2\pi T_1) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) . \tag{34.22}$$

To apply equations (34.21) we evaluate the above at $\omega = m\omega_1/2$, $\omega T_1/2 = m\pi/2$ so $\sin(m\pi/2) = 0$ for m even and $(-1)^{(m-1)/2}$ for m odd. Therefore,

$$X_{pulse}(m\omega_1/2) = T_1 (-1)^{(m-1)/2} / (m\pi/2) = (2/\pi)\, T_1 (-1)^{(m-1)/2} (1/m) = (4/\omega_1) (-1)^{(m-1)/2} (1/m)$$

$$\mathcal{P}_{pulse}(m\omega_1/2) = (1/\omega_1) (-1)^{(m-1)} / (m\pi/2)^2 = (2/\pi)^2 (1/\omega_1) (-1)^{(m-1)} (1/m^2) .$$

For m odd, $(-1)^{(m-1)} = 1$, so we get

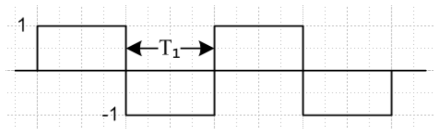


Fig 34.3

$$X(\omega) = 4 \sum_{m = \pm odd} (-1)^{(m-1)/2} (1/m)\delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega) = (2/\pi)^2 \sum_{m = \pm odd} (1/m^2)\, \delta(\omega - m\omega_1/2) \quad . \qquad (34.23)$$

<u>Square wave pulse train with peak-to-peak = 1 units and period $2T_1$</u>

In (34.23) X goes to 1/2 and $\mathscr{P}$ goes to 1/4 :

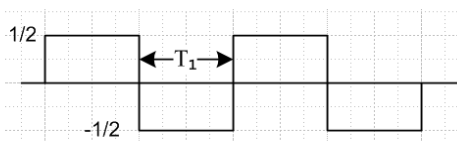


Fig 34.4

$$X(\omega) = 2 \sum_{m = \pm odd} (-1)^{(m-1)/2} (1/m)\delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega) = (1/\pi)^2 \sum_{m = \pm odd} (1/m^2)\, \delta(\omega - m\omega_1/2) \quad . \qquad (34.24)$$

<u>Square wave pulse train with peak-to-peak = 2 units and period T_1</u> :

In (34.23) replace $\omega_1 \rightarrow 2\omega_1$:

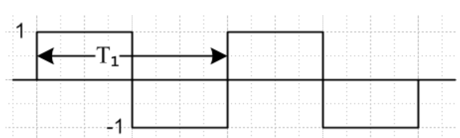


Fig 34.5

$$X(\omega) = 4 \sum_{m = \pm odd} (-1)^{(m-1)/2} (1/m)\delta(\omega - m\omega_1)$$

$$\mathscr{P}(\omega) = (2/\pi)^2 \sum_{m = \pm odd} (1/m^2)\, \delta(\omega - m\omega_1) \quad . \qquad (34.25)$$

<u>Square wave pulse train with peak-to-peak = 1 unit and period T_1</u> :

In (34.25) X goes to 1/2 and $\mathscr{P}$ goes to 1/4 :

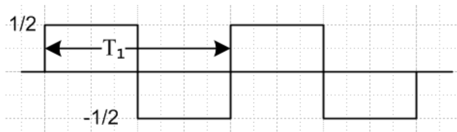


Fig 34.6

$$X(\omega) = 2 \sum_{m = \pm odd} (-1)^{(m-1)/2} (1/m)\delta(\omega - m\omega_1)$$

$$\mathcal{P}(\omega) = (1/\pi)^2 \sum_{m = \pm odd} (1/m^2)\, \delta(\omega - m\omega_1) \quad . \qquad (34.26)$$

In (17.9) we showed that for the above square wave $c_m = (1/\pi m)\, (i)^{1-m}$ for odd m. Then (17.4) says

$$X(\omega) = \sum_{m = \pm odd} c_m 2\pi\delta(\omega - m\omega_1) = \sum_{m = \pm odd} (1/\pi m)\, (i)^{1-m}\, 2\pi\delta(\omega - m\omega_1)$$

$$= 2 \sum_{m = \pm odd} (1/m)\, (-1)^{(m-1)/2}\, \delta(\omega - m\omega_1) \quad . \qquad \text{// agrees with (34.26)}$$

The power density from (33.27) is

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} |c_m|^2\, \delta(\omega - m\omega_1) = \sum_{m = \pm odd} (1/\pi m)^2\, \delta(\omega - m\omega_1) \quad . \qquad \text{// agrees with (34.26)}$$

Special Case 2 : Every other pulse is zero.

Here A = 1 and B = 0. Our general results (34.18) and (34.20) become

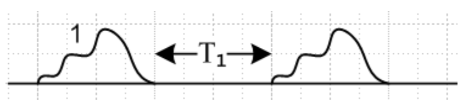


Fig 34.7

$$X(\omega) = X_{pulse}(\omega)\, (1/2)\, \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, (1/4)\, \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \quad . \qquad (34.27)$$

Inserting the same square wave pulse shown in (34.22) this becomes

$$X(\omega) = \pi \sum_{m=-\infty}^{\infty} \text{sinc}(m\pi/2)\, \delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega) = (1/4) \sum_{m=-\infty}^{\infty} \text{sinc}^2(m\pi/2)\, \delta(\omega - m\omega_1/2) \quad .$$

Separating out the m = 0 term and using $\sin(m\pi/2) = (-1)^{(m-1)/2}$ we get

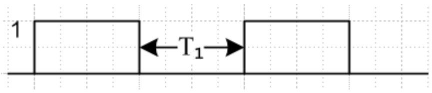


Fig 34.8

$$X(\omega) = 2 \sum_{m=\pm odd} (1/m)\ (-1)^{(m-1)/2}\ \delta(\omega - m\omega_1/2) + (1/2)\, 2\pi\delta(\omega)$$

$$\mathscr{P}(\omega) = (1/\pi^2) \sum_{m=\pm odd} (1/m^2)\ \delta(\omega - m\omega_1/2) + (1/4)\,\delta(\omega) \qquad (34.28)$$

These match (34.24) but here we have DC terms.

Special Case 3 : Recovering the Simple Pulse Train Spectra

Here A = 1 and B = 1. Our general results (34.18) and (34.20) become

$$X(\omega) = X_{pulse}(\omega)\ (1/2)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ 1 + (-1)^m\ ]\ \delta(\omega - m\omega_1/2) \qquad \textit{(34.18)}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\ (1/2)\ \omega_1 \sum_{m=-\infty}^{\infty} \{\ 1 + (-1)^m\ \}\ \delta(\omega - m\omega_1/2) \qquad \textit{(34.20)}$$

or

$$X(\omega) = X_{pulse}(\omega)\ \omega_1 \sum_{m=\pm even} \delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\ \omega_1 \sum_{m=\pm even} \delta(\omega - m\omega_1/2)$$

or

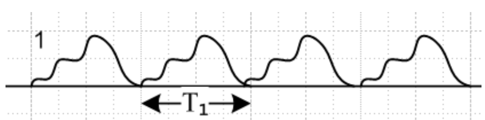


Fig 34.9

$$X(\omega) = X_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \qquad \text{// agrees with (14.5) for simple pulse train}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \qquad \text{// agrees with (33.26) for simple pulse train}$$

<u>Exercises for the Reader:</u>

(a) If the repeating amplitude sequence is A,B,C show that

$$X(\omega) = X_{pulse}(\omega)\,(1/3)[\,A + B\,e^{-i\omega T_1} + C\,e^{-i2\omega T_1}\,]\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/3)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,(1/3)^2\,|A + Be^{-i\omega T_1} + C\,e^{-i2\omega T_1}|^2\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/3)\;. \qquad (34.29)$$

(b) If the repeating sequence is $y_0, y_1 \ldots y_{P-1}$ show that

$$X(\omega) = X_{pulse}(\omega)\,\frac{1}{P}\,[\,\sum_{k=0}^{P-1} y_k e^{-ik\omega T_1}\,]\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\frac{1}{P^2}\,|\,\sum_{k=0}^{P-1} y_k e^{-ik\omega T_1}\,|^2\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)\;. \qquad (34.30)$$

These results can be expressed in terms of the Z Transform $Y_P''(z) = \Sigma_{k=0}^{P-1}\, y_k\, z^{-k}$ of the repeated sequence, where $z = e^{ik\omega T_1}$:

$$X(\omega) = X_{pulse}(\omega)\,\frac{1}{P}\;Y_P''(z)\;\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\frac{1}{P^2}\,|\,Y_P''(z)\,|^2\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \qquad (34.31)$$

<u>Hint</u>: These two equations are derived in Appendix F as (F.12) and (F.23).

35. Statistical Pulse Trains

(a) What is a Statistical Pulse Train?

Suppose we use a logic analyzer or digital scope to capture a 1 GHz PAM signal (zeros and ones) for 1 μsec so a sequence of K = 1,000 symbols y_m are stored. We could compute the average value for this sequence and we might find $\langle y_m\rangle_1 = 0.55$. Here $\langle ...\rangle_1$ refers to a **horizontal average** over a single pulse train amplitude sequence. By its definition, this average cannot depend on the index m, we are just computing an average of the y_m values in the sequence,

$$\langle y_m\rangle_1 = (1/K)\ \Sigma_{m=1}{}^{K}\ y_m\ .$$

Now it might be that during this 1 μsec, there was something unusual about the data being sent, so that this measurement of $\langle y_m\rangle_1 = 0.55$ is not really representative over a longer term. One alternative would be to capture a much longer sequence. Rather than do this, our approach will be to capture very many 1 μsec pulse trains and form an **ensemble** of these representative pulse trains. Perhaps we do this for a whole minute, so the ensemble then has a huge number (call it I) of pulse trains. We then write down these amplitude sequences in a vertical list on a very long piece of paper, one sequence below the next. We then number the positions in the sequences 1 to N and we then define $\langle y_m\rangle$ as the **vertical average** through this ensemble of the sequences in position m. This vertical average is written

$$\langle y_m\rangle = (1/I)\ \Sigma_{i=1}{}^{I}\ y_m{}^{(i)}$$

and in theory this could be different for different columns m. For example, consider this ensemble of sequences which has K = 6 and I = 3:

1	2	3	4	5	6	
a	b	c	d	e	f	sequence #1
a'	b'	c'	d'	e'	f'	sequence #2
a"	b"	c"	d"	d"	f"	sequence #3

Ensemble Fig 35.1

We would have

$\langle y_m\rangle_1 = (a+b+c+d+e+f)/6.$ // horizontal average for the first sequence

$\langle y_3\rangle = (c + c' + c")/3$ // vertical average for column 3

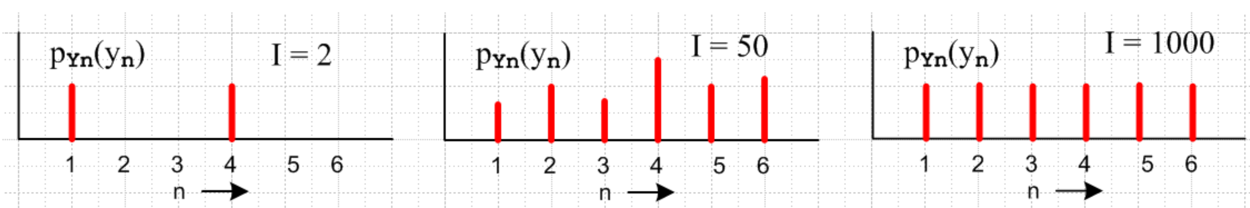
Our pulse train sequence has a **random variable** Y_m associated with each pulse train position, m = 1,2...K, and the vertical average $\langle y_m\rangle$ is the expected value of Y_m over the ensemble, normally written $\langle y_m\rangle = E(Y_m)$. The experiment associated with Y_m is the generation of a pulse train by some Apparatus, the experiment is run many times, and the outcomes for random variable Y_m are the values of the symbols located in position m of these pulse trains. See Appendix G for the meaning of "random variable".

The ensemble is characterized by various statistical properties, such as $E(Y_m)$ or $E(Y_nY_mY_k)$, each being a vertical average down through the ensemble. Our **statistical pulse train** is an idealized pulse train whose amplitudes form a **statistical sequence** whose statistics (like $E(Y_m)$ and $E(Y_nY_mY_k)$) exactly match those of the ensemble. It is not any particular member of the ensemble, since any member might deviate somehow. And it is certainly not the average of the pulse trains in the ensemble, since this average pulse train would have amplitudes y_m = <y_m> which has nothing to do with anything (and also would have illegal symbols, e.g., $y_m = 1/2$). If K were very large, a pulse train of the ensemble might come close to being a statistical pulse train.

Common usage refers to our statistical pulse train as a **random pulse train** whose sequence of amplitudes is a **random sequence**. The problem with this terminology is that the word "random" suggests for example that <y_m> = the mean value of Y_m in the ensemble = 1/2. That is to say, the word random suggests a flat distribution where p(1) = 1/2 and p(0) = 1/2. But if p(1) = 1/3 and p(0) = 2/3, our ensemble would still be associated with a statistical pulse train, Any value of $p \equiv p(1)$ would describe a statistical pulse train, along with the other statistical properties. This same subtle implication of the word random appears in Appendix G on "random variables" and basic probability theory.

In what follows, we shall only be concerned with the first and second order statistics of the statistical pulse train, which are <y_m> = $E(Y_m)$ and <$y_m y_n$> = $E(Y_m Y_n)$.

How many member pulse trains must the ensemble contain? Could it have only two pulse trains? The answer is that the ensemble could be small, but then the statistical properties of the random variables as discovered by the ensemble would have a large amount of error. For example, for column n there is some normalized distribution $p_{Yn}(y_n)$ associated with the random variable Y_n. Suppose y_n took values from {1,2,3,4,5,6}, and suppose $p_{Yn}(y_n)$ were a flat distribution. Here might be the different views of $p_{Yn}(y_n)$ as obtained from ensembles with I = 2, 50 and 1000 pulse trains :



Since $E(Y_n) = \Sigma_n\, y_n\ p_{Yn}(y_n)$, the error propagates into this and all other expectation values associated with the statistical pulse train. For any required degree of accuracy δ of the statistical quantities, we can find some value I_d such that for $I > I_\delta$ the ensemble will realize that accuracy. In theory, we just imagine that $I = \infty$, and then everything is perfect.

(b) The region of support for $y_m y_{m+s}$ for a finite pulse train

The product of amplitudes $y_m y_{m+s}$ appears many times below. For a finite pulse train it is useful to see where the product vanishes and where it does not vanish in terms of m and s. In the previous section we discussed a finite pulse train having y_n with n = 1 to K, but now we instead have y_n with n = -N to N so our finite pulses train now have (2N+1) amplitudes y_n. We imagine the finite pulse train embedded in an infinite pulse train that has all zeros to the left of -N and to the right of +N. In this case we may write for this infinite pulse train,

$$y_m y_{m+s} = y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(m+s \le N)\theta(m+s \ge -N)$$

$$= y_m y_{m+s}\, \theta(m \le N)\theta(m \ge -N)\, \theta(s \le N-m)\theta(s \ge -N-m) \qquad (35.1a)$$

where we use an inequality-style Heaviside step function θ. The product of the four θ functions and thus the quantity $y_m y_{m+s}$ vanishes outside the gray parallelogram in this drawing of the (m,s) plane,

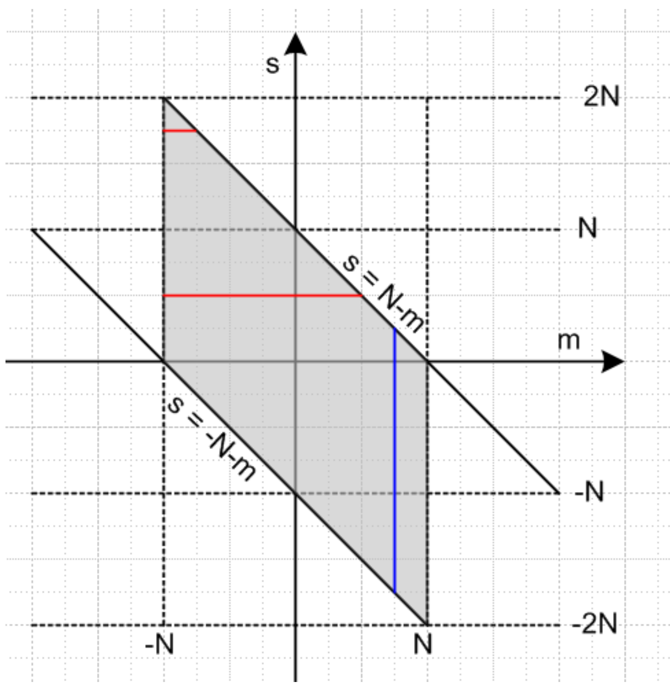


Fig 35.2

Thus, assuming m is in range -N to N, we may write

$$y_m y_{m+s} = \begin{cases} y_m y_{m+s} & \text{for } -N-m \le s \le N-m \\ 0 & \text{for } s < -N-m \text{ or } s > N-m \end{cases} \qquad (35.1b)$$

Below we shall be interested in the following quantity (the autocorrelation sequence),

$$r_s = \langle y_m y_{m+s} \rangle_1 = \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s} \qquad \text{// a horizontal average} \qquad (35.2)$$

$$= \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s}\ \theta(m \le N)\theta(m \ge -N)\ \theta(s \le N-m)\theta(s \ge -N-m)$$

$$= \frac{1}{2N+1} \sum_{m=-\infty}^{\infty} y_m y_{m+s}\ \theta(m \le N)\theta(m \ge -N)\ \theta(m \le N-s)\theta(m \ge -N-s)$$

$$= \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} y_m y_{m+s} \quad . \tag{35.3}$$

The range of m appearing in this sum is represented by a red line in the figure for two positive values of s. For s > 2N there is no range left so $r_s = 0$, and the same when s < -2N. Thus, we may regard

$$r_s = r_s\ \theta(s \le 2N)\theta(s \ge -2N)\ . \tag{35.4}$$

Another quantity of interest, R"(z) = Z Transform of r_s, therefore has this restricted sum range for a finite pulse train,

$$R''(z) = \sum_{s=-\infty}^{\infty} r_s\ z^{-s} = \sum_{s=-2N}^{2N} r_s\ z^{-s} \quad . \tag{35.5}$$

Notice that $\langle y_m y_{m+s}\rangle_1 = r_s$ does not depend on m due to its "horizontal" average definition shown in (35.2). The non-dependence of $\langle y_m y_{m+s}\rangle_1$ on m has nothing to do with "stationarity" as will be discussed below. That is, $\langle y_m y_{m+s}\rangle_1$ is for a particular pulse train, it is not an ensemble average.

(c) The Spectral Power Density for a specific pulse train

There are *many ways* to write the power density expression for a pulse train, see (32.18) and (34.14a). For an infinite pulse train we may write (reader should perhaps ponder each form),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} r_s z^{-s}$$

$$= \mathcal{P}_{pulse}(\omega)\, R''(z) \quad = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} |\, Y''(z)\,|^2 \qquad r_s = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} y_m y_{m+s}$$

$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} y_m z^{-m} \right)^* \left(\sum_{n=-\infty}^{\infty} y_n z^{-n} \right)$$

$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_m y_n z^{m-n} \right) \qquad // z = e^{i\omega T_1} \text{ so } (z^{-m})^* = z^m$$

$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} y_m y_{m+s} z^{-s} \right) \qquad \mathcal{P}(\omega) \text{ INFINITE} \qquad (35.6)$$

Recall that T is the duration of the infinite pulse train, written as $T = 2\pi\delta(0)$ in (33.22). This T always cancels away, so we are not concerned about its infinite nature. It is a limit $(2N+1)T_1$ as $N\rightarrow\infty$.

For a finite pulse train we find, using (35.3) and (35.5) and Fig 35.2,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} \langle y_m y_{m+s} \rangle_1 z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} r_s z^{-s}$$

$$= \mathcal{P}_{pulse}(\omega)\, R''(z) \quad = \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} |\, Y''(z)\,|^2 \qquad r_s = \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} y_m y_{m+s}$$

$$= \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \left(\sum_{m=-N}^{N} y_m z^{-m} \right)^* \left(\sum_{n=-N}^{N} y_n z^{-n} \right)$$

$$= \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \left(\sum_{m=-N}^{N} \sum_{n=-N}^{N} y_m y_n z^{m-n} \right)$$

$$= \mathcal{P}_{pulse}(\omega) \frac{1}{2N+1} \left(\sum_{m=-N}^{N} \sum_{s=-N-m}^{N-m} y_m y_{m+s} z^{-s} \right) \qquad \mathcal{P}(\omega) \text{ FINITE} \qquad (35.7)$$

In the last expression, the sum on s is represented by the blue line in Fig 35.2. In all these expressions, one can regard $y_m y_{m+s}$ and $y_m\, y_n$ as being stripped of their Heaviside θ functions as in (35.3). That is to say, there are no places in any of the above expressions where $y_m y_{m+s} = 0$ because it is out of range. Later when we add stationarity, we will replace $y_m\, y_{m+s}$ by f(s) which is never 0 in an out-of-range sense.

(d) The mean Spectral Power Density for an Ensemble of pulse trains

We now apply our ensemble average <....> to the previous equation groups to get for the infinite case,

$$
\begin{aligned}
<\mathscr{P}(\omega)> &= \mathscr{P}_{\mathtt{pulse}}(\omega) \sum_{s=-\infty}^{\infty} <<y_m y_{m+s}>_1> z^{-s} = \mathscr{P}_{\mathtt{pulse}}(\omega) \sum_{s=-\infty}^{\infty} <r_s> z^{-s} \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) <R''(z)> = \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{T_1}{T} < | Y''(z) |^2> \qquad <r_s> = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} <y_m y_{m+s}> \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{T_1}{T} < (\sum_{m=-\infty}^{\infty} y_m z^{-m})^* (\sum_{n=-\infty}^{\infty} y_n z^{-n}) > \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{T_1}{T} (\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} <y_m y_n> z^{m-n}) \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{T_1}{T} (\sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} <y_m y_{m+s}> z^{-s}) \qquad <\mathscr{P}(\omega)> \text{ INFINITE} \qquad (35.8)
\end{aligned}
$$

and for an ensemble of finite pulse trains,

$$
\begin{aligned}
<\mathscr{P}(\omega)> &= \mathscr{P}_{\mathtt{pulse}}(\omega) \sum_{s=-2N}^{2N} <<y_m y_{m+s}>_1> z^{-s} = \mathscr{P}_{\mathtt{pulse}}(\omega) \sum_{s=-2N}^{2N} <r_s> z^{-s} \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) <R''(z) > = \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{1}{2N+1} <| Y''(z) |^2>, \qquad <r_s> = \frac{1}{2N+1} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} <y_m y_{m+s}> \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{1}{2N+1} < (\sum_{m=-N}^{N} y_m z^{-m})^* (\sum_{n=-N}^{N} y_n z^{-n}) > \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{1}{2N+1} (\sum_{m=-N}^{N} \sum_{n=-N}^{N} <y_m y_n> z^{m-n}) \\
&= \mathscr{P}_{\mathtt{pulse}}(\omega) \frac{1}{2N+1} (\sum_{m=-N}^{N} \sum_{s=-N-m}^{N-m} <y_m y_{m+s}> z^{-s}) \qquad <\mathscr{P}(\omega)> \text{ FINITE} \qquad (35.9)
\end{aligned}
$$

(e) Adding Stationarity to the Ensemble Situation

The stationarity assumption is that the Apparatus generating the pulse trains of the ensemble is a "**stationary** stochastic process" which means (at least) that

$<y_m>$ does not depend on m // where $<y_m>$ does not vanish
$<y_m y_{m+s}>$ does not depend on m // where $<y_m y_{m+s}>$ does not vanish (35.10)

Recall that for a finite pulse train, $<y_m>$ vanishes outside the range (-N,N) while $<y_m y_{m+s}>$ vanishes outside the gray region in Fig 35.2. For an infinite pulse train these objects don't vanish anywhere.

From Appendix G we know that

$$\mathrm{cov}(X,Y) = E(XY) - \mu_x\mu_y \qquad \textit{(G.24b)}$$

$$\Rightarrow \mathrm{cov}(Y_m, Y_{m+s}) = \langle y_m y_{m+s}\rangle - \langle y_m\rangle\langle y_{m+s}\rangle = \langle y_m y_{m+s}\rangle - \langle y_m\rangle^2 \quad \text{// stationarity of } \langle y_m\rangle$$

Assuming $\langle y_m\rangle$ and $\langle y_m y_{m+s}\rangle$ are stationary is the same as assuming $\langle y_m\rangle$ and cov (Y_m, Y_{m+s}) are stationary. If we make no stationarity requirement on other statistical measures (like $\langle y_m^3\rangle$), this limited sense of stationarity is usually called **wide-sense stationarity** (WSS).

This stationarity assumption applies to both finite and infinite pulse trains. It is an assumption about the process which generates pulse trains regardless of their length.

It is helpful to define $\mu \equiv \langle y_m\rangle$ and $f(s) \equiv \langle y_m y_{m+s}\rangle$ and rewrite the stationarity assumption in this manner for a finite pulse train, where the second equation invokes Fig 35.2's gray region,

$$\langle y_m\rangle = \begin{cases} \mu & \text{for } -N \le m \le N \\ 0 & \text{for } m < -N \text{ or } m > N \end{cases}$$

$$\langle y_m y_{m+s}\rangle = \begin{cases} f(s) & \text{for } -N-m \le s \le N-m \\ 0 & \text{for } s < -N-m \text{ or } s > N-m \end{cases} \qquad \text{// m in (-N,N)} \qquad (35.11)$$

For an infinite pulse train, the upper lines apply and then $\langle y_m\rangle = \mu$ and $\langle y_m y_{m+s}\rangle = f(s)$ everywhere.

With this stationarity assumption, we can write

$$\langle r_s\rangle = \frac{T_1}{T}\sum_{m=-\infty}^{\infty}\langle y_m y_{m+s}\rangle = \frac{T_1}{T} f(s) \sum_{m=-\infty}^{\infty}[1] = f(s) \qquad \text{// infinite pulse train}$$

$$\langle r_s\rangle = \frac{1}{2N+1}\sum_{m=\max(-N,-N-s)}^{\min(N,N-s)}\langle y_m y_{m+s}\rangle = \frac{1}{2N+1} f(s) \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)}[1] =$$

$$= \frac{1}{2N+1} f(s)\,(2N+1-|s|) \qquad \text{// see red lines in Fig 35.2}$$

$$= \frac{2N+1-|s|}{2N+1} f(s) \qquad \text{// finite pulse train}$$

To summarize :

$$\langle r_s\rangle = f(s) \qquad \text{// infinite pulse train}$$

$$\langle r_s\rangle = \frac{2N+1-|s|}{2N+1} f(s) \qquad \text{// finite pulse train} \qquad (35.12)$$

Here then are the infinite pulse train expressions assuming stationarity :

$$<\mathscr{P}(\omega)> = \mathscr{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} <<y_m y_{m+s}>_1> z^{-s} = \mathscr{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} <r_s> z^{-s}$$

$$= \mathscr{P}_{pulse}(\omega) <R''(z)> = \mathscr{P}_{pulse}(\omega) \frac{T_1}{T} <|Y''(z)|^2> \qquad <r_s> = f(s)$$

$$= \mathscr{P}_{pulse}(\omega) \frac{T_1}{T} <(\sum_{m=-\infty}^{\infty} y_m z^{-m})^* (\sum_{n=-\infty}^{\infty} y_n z^{-n})>$$

$$= \mathscr{P}_{pulse}(\omega) \frac{T_1}{T} (\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} <y_m y_n> z^{m-n})$$

$$= \mathscr{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} f(s) z^{-s} \qquad <\mathscr{P}(\omega)> \text{ INFINITE (stat)} \qquad (35.13)$$

In the last line we used

$$\frac{T_1}{T} (\sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} <y_m y_{m+s}> z^{-s}) = \frac{T_1}{T} (\sum_{s=-\infty}^{\infty} f(s) z^{-s}) \sum_{m=-\infty}^{\infty} [1] = \sum_{s=-\infty}^{\infty} f(s) z^{-s}$$

and then the last line just replicates the first line given that $<r_s> = f(s)$.

And here are the *finite* pulse train $<\mathscr{P}(\omega)>$ expressions assuming stationarity :

$$<\mathscr{P}(\omega)> = \mathscr{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} <<y_m y_{m+s}>_1> z^{-s} = \mathscr{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} <r_s> z^{-s}$$

$$= \mathscr{P}_{pulse}(\omega) <R''(z)> = \mathscr{P}_{pulse}(\omega) \frac{1}{2N+1} <|Y''(z)|^2> \qquad <r_s> = \frac{2N+1-|s|}{2N+1} f(s)$$

$$= \mathscr{P}_{pulse}(\omega) \frac{1}{2N+1} <(\sum_{m=-N}^{N} y_m z^{-m})^* (\sum_{n=-N}^{N} y_n z^{-n})>$$

$$= \mathscr{P}_{pulse}(\omega) \frac{1}{2N+1} (\sum_{m=-N}^{N} \sum_{n=-N}^{N} <y_m y_n> z^{m-n})$$

$$= \mathscr{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} f(s) \frac{2N+1-|s|}{2N+1} z^{-s} \qquad <\mathscr{P}(\omega)> \text{ FINITE (stat)} \qquad (35.14)$$

In the last line we did this manipulation on the last line of (35.9) (see the gray region in Fig 35.2)

$$\frac{1}{2N+1} \sum_{m=-N}^{N} \sum_{s=-N-m}^{N-m} <y_m y_{m+s}> z^{-s} = \frac{1}{2N+1} \sum_{s=-2N}^{2N} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} f(s) z^{-s}$$

$$= \frac{1}{2N+1} \sum_{s=-2N}^{2N} f(s) z^{-s} \sum_{m=\max(-N,-N-s)}^{\min(N,N-s)} [1] = \frac{1}{2N+1} \sum_{s=-2N}^{2N} f(s) z^{-s} (2N+1-|s|)$$

One again, we see that the final expression in (35.14) replicates the first one with $\langle r_s \rangle$ as shown.

(f) Adding Independence along with Stationarity to the Ensemble Situation

As shown in Appendix G, if Y_n and Y_m are independent random variables, then for $s \neq 0$,

$$f(s) = \langle y_m\, y_{m+s} \rangle = E(Y_n Y_{m+s}) = E(Y_n)E(Y_{m+s}) = \langle y_m \rangle \langle y_{m+s} \rangle = \langle y_m \rangle \langle y_m \rangle = \langle y_m \rangle^2 \quad \text{// stationarity}$$

or

$$f(s) = \mu^2 \qquad\qquad \mu = \langle y_m \rangle$$

But for s = 0, the result is different. We write

$$f(0) = \langle y_m y_m \rangle = \langle y_m^2 \rangle = \sigma^2 + \mu^2 \qquad \text{where } \sigma^2 \equiv \langle y_m^2 \rangle - \langle y_m \rangle^2 = \langle y_m^2 \rangle - \mu^2 \qquad \textit{(G.30)}$$

To summarize:

$$f(s) = \begin{cases} \mu^2 & s \neq 0 \\ \sigma^2 + \mu^2 & s = 0 \end{cases} . \qquad \text{// stationarity and independence assumed} \qquad (35.15)$$

For the last time (!), we consider the various $\langle \mathcal{P}(\omega) \rangle$ expressions now with stationarity and independence assumed. Rather than write all the expressions, we focus just on the first equation (same as the last) of the group (35.13) for the infinite case,

$$\begin{aligned}
\langle \mathcal{P}(\omega) \rangle &= \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle r_s \rangle\, z^{-s} = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} f(s)\, z^{-s} \\
&= \mathcal{P}_{pulse}(\omega)\, [\, f(0) + \sum_{s \neq 0} f(s)\, z^{-s}\,] \\
&= \mathcal{P}_{pulse}(\omega)\, [(\sigma^2 + \mu^2) + \mu^2 \sum_{s \neq 0} z^{-s}\,] \\
&= \mathcal{P}_{pulse}(\omega)\, [\sigma^2 + \mu^2 \sum_{s=-\infty}^{\infty} z^{-s}\,] \qquad \text{// now use (13.2) with } z = e^{i\omega T_1} \text{ to get :} \\
&= \mathcal{P}_{pulse}(\omega)\, [\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\,] . \qquad \langle \mathcal{P}(\omega) \rangle \text{ INFINITE (stat+indep)} \qquad (35.16)
\end{aligned}$$

We shall return to this classic result below. Meanwhile, we have from the first equation (same as the last) of the group (35.14) for the *finite* case,

$$\begin{aligned}
\langle \mathcal{P}(\omega) \rangle &= \mathcal{P}_{pulse}(\omega) \sum_{s=-2N}^{2N} f(s)\, z^{-s}\, \frac{2N+1-|s|}{2N+1} \\
&= \mathcal{P}_{pulse}(\omega) [\, f(0) + \sum_{s \neq 0} f(s)\, z^{-s}\, \frac{2N+1-|s|}{2N+1}\,]
\end{aligned}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)[\ (\sigma^2 + \mu^2)\ +\ \mu^2 \sum_{s \neq 0} \frac{2N+1 - |s|}{2N+1}\ z^{-s}\ ]$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)[\ \sigma^2\ +\ \mu^2 \sum_{s=-2N}^{2N} \frac{2N+1 - |s|}{2N+1}\ z^{-s}] \qquad \text{// see App. D and comments below}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)[\ \sigma^2\ +\ \mu^2 \frac{1}{2N+1} \frac{\sin^2[(N+1/2)\omega T_1]}{\sin^2(\omega T_1/2)}\ ] \qquad \text{// now use (A.20) to get:}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)[\ \sigma^2\ +\ \mu^2\, 2\pi\, \delta_6(\omega T_1,N)\ ]\ . \qquad \mathbf{<}\mathcal{P}(\omega)\mathbf{>}\ \text{FINITE (stat+indep)} \qquad (35.17)$$

The evaluation of the Σ_s in the third last line is non-trivial and is carried out in Appendix D. The last line expresses the second last line in the language of Appendix A where δ_6 is the delta function model shown in (A.20). From (A.21) we know that

$$\lim_{N\to\infty} \delta_6(k,N)\ =\ \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad \textit{(A.21)}$$

so the last line of (35.17) becomes in the limit N→∞,

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega)[\ \sigma^2\ +\ \mu^2\, 2\pi \sum_{m=-\infty}^{\infty} \delta(k-2\pi m)\ ]$$

which (as expected) agrees with the last line of (35.16) for the infinite case.

Evaluation of the Σ_s in (3.17) is not really necessary since we can compute $<\mathcal{P}(\omega)>$ using a different equation from the group (35.13), namely

$$<\mathcal{P}(\omega)>\ = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \frac{1}{2N+1}\ (\ \sum_{m=-N}^{N} \sum_{n=-N}^{N} <y_m\, y_n>\ z^{m-n}\ )\ . \qquad (35.18)$$

Sometimes double sums (or double integrals) are easier to evaluate than a single-sum representation of the same function. The double sum in (35.18) may be evaluated as follows, using (35.15),

$$\sum_{n=-N}^{N} \sum_{m=-N}^{N} <y_m\, y_n>\ z^{n-m}\ =\ \sum_{n=-N}^{N} [\ \sum_{m=n} f(0)\, z^{n-m}\ +\ \sum_{m\neq n} f(n-m)\, z^{n-m}\ ]$$

$$= \sum_{n=-N}^{N} [(\sigma^2 + \mu^2)\ + \mu^2 \sum_{m\neq n} z^{n-m}\ +] \qquad \text{// in first term } \sum_{m=n} z^{n-m}\ = 1$$

$$= \sum_{n=-N}^{N} [\ \sigma^2\ +\ \mu^2 \sum_{m=-N}^{N} z^{n-m}\ +]$$

$$= (2N+1)\, \sigma^2 + \mu^2\ (\ \sum_{n=-N}^{N} z^n)(\ \sum_{m=-N}^{N} z^m)^*$$

$= (2N+1)\,\sigma^2 + \mu^2 \;\; [\, 2\pi\, \delta_5(\omega T_1,N)\,]^2$ // from (13.3) used twice with $k = \omega T_1$

$= (2N+1)\,\sigma^2 + \mu^2 \;\; [\, (2N+1)\, 2\pi\, \delta_6(\omega T_1,N)\,]$ // from (A.20) (35.19)

and therefore

$$<\mathscr{P}(\omega)> \;=\mathscr{P}_{pulse}(\omega)\, \frac{1}{2N+1} \;\{\; (2N+1)\,\sigma^2 + \mu^2\, [\, (2N+1)\, 2\pi\, \delta_6(\omega T_1,N) \quad]\,\}$$

$$=\mathscr{P}_{pulse}(\omega)[\,\sigma^2 \;+\; \mu^2\, 2\pi\, \delta_6(\omega T_1,N)]$$

which agrees with the last line of (35.17), and we bypassed doing the Appendix D sum.

(g) Comparison between the two kinds of averages <..> and <...>$_1$

We now want to compare the two averages appearing in the above equations :

$<y_m>_1$ cannot depend on m from its definition as a horizontal average over a sequence.
$<y_m>$ might depend on m, but does not with the stationarity assumption

$<y_m y_{m+s}>_1 = r_s$ cannot depend on m from its definition (32.17), but generally depends on s
$<y_m y_{m+s}>$ may depend on m as well as s, but with stationarity depends only on s

$<y_m y_{m+s}>_1$ is not associated with any random variables
$<y_m y_{m+s}>$ is associated with random variables Y_m and Y_{m+s} as outlined in section (a) above

If Y_m and Y_{m+s} are independent random variables, then $<y_m y_{m+s}> \;=\; <y_m><y_{m+s}>$ (see App G).
In contrast, the statement $<y_m y_{m+s}>_1 \;=\; <y_m>_1 < y_{m+s}>_1 \;= <y_m>_1 < y_m>_1 \;= < y_m>_1^{\,2}$ might be true, but has no connection to any random variables upon whose independence this factoring could be postulated.

(35.20)

In order to make a connection between these two kinds of averages, we have to make some assumptions.

For an ensemble of pulse trains of *very long* length (large N), we assume that each ensemble member is a "statistical pulse train" in that it very closely has the statistics of the ensemble as a whole. This does not mean that the ensemble pulse trains are the same.

One implication of this assumption is that all member pulse trains (i = 1,2.. I) then have the same autocorrelation function and we may then write

$r_s^{(i)} = \langle y_m y_{m+s}\rangle_1^{(i)} = (1/N)\,\Sigma_{m=1}^N\, y_m^{(i)}\, y_{m+s}^{(i)}$ = the same for all i = 1 to I.

$$\equiv r_s \equiv \langle y_m y_{m+s}\rangle_1 \quad . \tag{35.21}$$

Consider then,

$\Sigma_{m=1}^N \langle y_m y_{m+s}\rangle = N \langle y_m y_{m+s}\rangle$. // stationarity

On the left side of this equation we insert the definition of $\langle y_m y_{m+s}\rangle$,

$$\langle y_m y_{m+s}\rangle = (1/I)\,\Sigma_{i=1}^I\, y_m^{(i)}\, y_{m+s}^{(i)} , \tag{35.22}$$

with this result

$\Sigma_{m=1}^N (1/I)\,\Sigma_{i=1}^I\, y_m^{(i)}\, y_{m+s}^{(i)} = N \langle y_m y_{m+s}\rangle$.

Now reorder the sums to get,

$(N/I)\,\Sigma_{i=1}^I [(1/N)\,\Sigma_{m=1}^N\, y_m^{(i)} y_{m+s}^{(i)}] = N \langle y_m y_{m+s}\rangle$

$(N/I)\,\Sigma_{i=1}^I \langle y_m y_{m+s}\rangle_1^{(i)} = N \langle y_m y_{m+s}\rangle$

$(1/I)\,\Sigma_{i=1}^I \langle y_m y_{m+s}\rangle_1 = \langle y_m y_{m+s}\rangle$ // use (35.21) and cancel the N's

$\langle y_m y_{m+s}\rangle_1 [\, (1/I)\,\Sigma_{i=1}^I 1\,] = \langle y_m y_{m+s}\rangle$

$\langle y_m y_{m+s}\rangle_1 = \langle y_m y_{m+s}\rangle$. // horizontal average equals vertical average

As a special case, we can set s = 0 to get

$\langle y_m^2\rangle_1 = \langle y_m^2\rangle$.

In similar fashion, our very large N assumption implies that all pulse trains in the ensemble have the same horizontal mean, so

$$\langle y_m\rangle_1^{(i)} \equiv \langle y_m\rangle_1 \quad . \tag{35.23}$$

Repeating the above steps now for $\langle y_m \rangle$,

$\Sigma_{m=1}^N \langle y_m \rangle = N \langle y_m\rangle$ // stationarity

$\Sigma_{m=1}^N [\, (1/I)\,\Sigma_{i=1}^I\, y_m^{(i)}] = N \langle y_m\rangle$

$(N/I)\,\Sigma_{i=1}^I (1/N)\,\Sigma_{m=1}^N [y_m^{(i)}] = N \langle y_m\rangle$

$(N/I)\ \Sigma_{i=1}^{I} <y_m>_1^{(i)} \ = \ N\ <y_m>$

$(1/I)\ \Sigma_{i=1}^{I} <y_m>_1 \ = <y_m>$ // use (35.23) and cancel the N's

$<y_m>_1 \ = <y_m>$.

Thus, assuming stationarity and very large N, we have found that

$$<y_m y_{m+s}>_1 \ = <y_m y_{m+s}>$$
$$<y_m y_m>_1 \ = <y_m y_m>$$
$$<y_m>_1 \ = <y_m> \ . \qquad \text{// all three lines only for very large N} \qquad (35.24)$$

These relations are equalities for N = ∞, are approximately valid for large N, and are invalid for small N.

If we add to our assumptions so far that Y_m and Y_{m+s} are independent, not only may we write

$$E(Y_m Y_{m+s}) = E(Y_m)E(Y_{m+s}) \quad \Leftrightarrow \quad <y_m y_{m+s}> = <y_m>< y_{m+s}> \qquad \text{// independent, } s\neq 0$$

but we may also write, using stationarity and independence

$$<y_m y_{m+s}> = <y_m>< y_m> \ = <y_m>^2 \qquad s \neq 0$$

which by (35.24) then justifies the claim sometimes made that,

$$<y_m y_{m+s}>_1 = <y_m>_1 < y_m>_1 \ = <y_m>_1^2 \quad s\neq 0 \qquad \text{//stationary + independent + N}\to\infty \qquad (35.25)$$

(h) Miscellaneous topics

The Two Methods for computing $\mathcal{P}(\omega)$

From (35.6) and (35.8) we have obtained these two general "formulas" for the spectral power density of a an infinite pulse train, prior to assuming stationarity or independence. We first write these as

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left(\sum_{s=-\infty}^{\infty} r_s z^{-s} \right) \qquad \text{first line of (35.6) "autocorrelation"}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} y_m z^{-m} \right)^* \left(\sum_{n=-\infty}^{\infty} y_n z^{-n} \right) \qquad (35.26)$$
$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_m y_n z^{m-n} \right) \qquad \text{third line of (35.6) "double sum"}$$

The corresponding equations for finite pulse trains are,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left(\sum_{s=-2N}^{2N} r_s z^{-s} \right) \qquad \text{first line of (35.7) "autocorrelation"}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-N}^{N} y_m z^{-m} \right)^* \left(\sum_{n=-N}^{N} y_n z^{-n} \right) \tag{35.26)'$$

$$= \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{m=-N}^{N} \sum_{n=-N}^{N} y_m y_n z^{m-n} \right) \qquad \text{third line of (35.7) "double sum"}$$

In either case we have two distinct methods for the computation of $\mathcal{P}(\omega)$. We might call these the **autocorrelation method** which uses r_s, and the **double-sum method** which directly uses the y_n data without the intermediary of r_s. We saw earlier that the double-sum method provided an easier pathway for the finite pulse train case, leading to the last line of (35.17) using the steps shown in (35.19), but in general the autocorrelation method is the one most commonly used.

The autocorrelation method will be used in Section 37 for the AMI line code, in Section 38 for the Change/Hold Line code, and in Appendix F for infinite pulse trains with repeated subsequences such as the Maximum Length Sequence output by a shift register generator.

The equivalence of $\mathcal{P}(\omega)$ and $\langle\mathcal{P}(\omega)\rangle$ for infinite pulse trains

We now rewrite (35.26), this time using an ensemble average on the lower equation,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left(\sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} \right) \qquad \text{one pulse train} \tag{\textit{35.6}}$$

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{pulse}(\omega) \frac{T_1}{T} \left(\sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \right) \qquad \text{ensemble of pulse trains} \tag{\textit{35.8}}$$

When stationarity is assumed for the process creating the pulse trains, in the second line above we may slide $\langle y_m y_{m+s}\rangle z^{-s}$ to the left through the Σ_m sum, which sum becomes $\Sigma_m[1] = T/T_1$ and we then have this pair of formulas

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle_1 z^{-s} = \mathcal{P}_{pulse}(\omega) \left(\sum_{s=-\infty}^{\infty} r_s z^{-s} \right) \qquad \text{one pulse train} \quad \textit{(35.7)}$$

$$\langle\mathcal{P}(\omega)\rangle = \mathcal{P}_{pulse}(\omega) \sum_{s=-\infty}^{\infty} \langle y_m y_{m+s} \rangle z^{-s} \quad . \qquad \text{ensemble of pulse trains} \quad \textit{(35.8)}$$

(35.27)

Our point is merely that, according to (35.24) which states $\langle y_m y_{m+s}\rangle_1 = \langle y_m y_{m+s}\rangle$, the two right hand side expressions are the *same* for infinite pulse trains, and <u>there is no distinction</u> between $\langle\mathcal{P}(\omega)\rangle$ for the ensemble and $\mathcal{P}(\omega)$ for any member of the ensemble. There is also no distinction between r_s and $\langle r_s\rangle$ since every pulse train in the ensemble has the same autocorrelation function as in (35.21).

Facts about Infinite Uncorrelated Statistical Pulse Trains

Here we use the word "uncorrelated" as a sort of euphemism for what is really a stationary and independent statistical pulse train as described above. It is true that independent does imply uncorrelated as shown in Appendix G (c). Since the converse is not true, our term is somewhat inaccurate.

In the special case of stationarity and independence, we found that from (35.12),

$$r_s = \langle r_s \rangle = f(s) = \begin{cases} \mu^2 & s \neq 0 \\ \sigma^2 + \mu^2 & s = 0 \end{cases} \qquad \langle y_m \rangle = \mu, \quad \langle y_m^2 \rangle = \sigma^2 + \mu^2 \tag{35.12}$$

where

$$r_s = \lim_{N\to\infty} \left[\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s} \right] . \tag{32.16}$$

A plot of r_s is usually presented in this traditional manner,

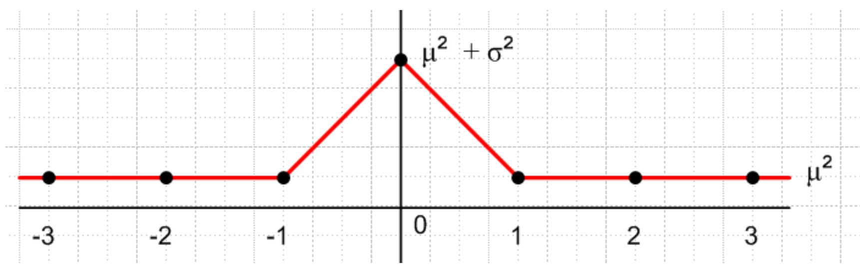


Fig 35.3

and the corresponding ensemble average power spectrum was shown in the last line of (35.16)

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{pulse}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right] . \tag{35.16}$$

The spectrum has a continuous piece $\sigma^2\, \mathcal{P}_{pulse}(\omega)$ proportional to the variance σ^2 of the y_n over the ensemble, and a discrete set of lines proportional to the square of the mean $\mu = \langle y_m \rangle$ of the ensemble. According to the comments above and (35.24), we can think of $\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$ of any *member* of the ensemble, $\mu = \langle y_m \rangle = \langle y_m \rangle_1$ as the mean for that member, and similarly $\sigma^2 = \langle y_m^2 \rangle - \mu^2 = \sigma^2 = \langle y_m^2 \rangle_1 - \mu^2$ as the variance for that member of the ensemble. With this reinterpretation of μ^2 and σ^2 as properties of a single pulse train, we then have for any member of the ensemble,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \right] . \tag{35.28}$$

Since

$$2\pi\delta(\omega T_1 - 2\pi m) = (2\pi/T_1)\, \delta(\omega - 2\pi m/T_1) = \omega_1\, \delta(\omega - m\omega_1) = \delta(\frac{\omega}{\omega_1} - m) \qquad (35.29)$$

we can write these alternate forms for (35.28) :

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, [\sigma^2 + \omega_1\, \mu^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\,] \qquad (35.28a)$$

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{pulse}(\omega) + \omega_1\, \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{pulse}(m\omega_1)\delta(\omega - m\omega_1) \qquad (35.28b)$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, [\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} \delta(\frac{\omega}{\omega_1} - m)\,] \qquad (35.28c)$$

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{pulse}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{pulse}(m\omega_1)\, \delta(\frac{\omega}{\omega_1} - m) \qquad (35.28d)$$

<u>Verification of (35.28)</u> // "trust but verify"

To find external verification for (35.28), we write the expression in terms of f where $\omega = 2\pi f$,

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = \frac{|\mathcal{X}_{pulse}(f)|^2}{2\pi T_1} \qquad \text{// (34.4) and text after (1.4)}$$

$$P(f) = 2\pi\mathcal{P}(\omega) \qquad \text{// (34.4) last item}$$

$$2\pi\, \delta(\omega T_1 - 2\pi m) = 2\pi\, \delta(2\pi f T_1 - 2\pi m) = (1/T_1)\, \delta(f - m/T_1) \ .$$

Then (35.28) becomes

$$\frac{1}{2\pi} P(f) = \frac{|\mathcal{X}_{pulse}(f)|^2}{2\pi T_1} \left(\sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(f - m/T_1) \right)$$

or

$$P(f) = \frac{|\mathcal{X}_{pulse}(f)|^2}{T_1} \left(\sigma^2 + \frac{\mu^2}{T_1} \sum_{m=-\infty}^{\infty} \delta(f - m/T_1) \right) \ . \qquad (35.28e)$$

This may be compared to result (A.17) in Appendix A of Xiong which we quote,

$$\Psi_s(f) = \frac{|G(f)|^2}{T} \left(\sigma_a^2 + \frac{m_a^2}{T} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T}) \right) \qquad (A.17)$$

QED

Specialization for symbols in the set {A,B}

As a simple example, suppose the set of symbols that our y_i can take is just {A,B}. Then the pmf distribution associated with random variable Y_n (see Appendix G) is quite simple :

$$p_{Yn}(x) = p \qquad \text{if } x = A$$
$$p_{Yn}(x) = 1\text{-}p \qquad \text{if } x = B \quad . \tag{35.30}$$

We can then compute

$$\mu = \langle y_m \rangle = [p]A + [1\text{-}p]B \, . \tag{35.31}$$

For $n \neq m$ we then have

$$\langle y_m y_n \rangle = \langle y_m \rangle^2 = \mu^2 = \{ pA + (1\text{-}p)B \}^2$$

$$= [pp]\, AA + [p(1\text{-}p)]\, AB + [(1\text{-}p)p]BA + [(1\text{-}p)(1\text{-}p)]BB \tag{35.32}$$

and for n = m,

$$\langle y_m^2 \rangle = [p]AA + [(1\text{-}p)]\, BB \, . \tag{35.33}$$

In the square brackets [..] we indicate the probability of some case occurring, and this is multiplied by the value that the quantity in question takes in that case. Notice how the long expression for $\langle y_m y_n \rangle$ makes complete sense if one enumerates all possibilities. The variance may be computed as

$$\sigma^2 = \langle y_n^2 \rangle - \mu^2 = [p]AA + [(1\text{-}p)]\, BB - \{ pA + (1\text{-}p)B \}^2$$

$$= [p(1\text{-}p)]\, (A\text{-}B)^2 \tag{35.34}$$

where Maple helps out,

```
p*A^2 + (1-p)*B^2 - ( p*A + (1-p)*B)^2: factor(%);
```

$$-p\,(A-B)^2\,(-1+p)$$

.

With the reinterpretations of μ^2 and σ^2 noted above (now applying to a single pulse train), we then have for our infinite statistical pulse train with independent amplitudes in the set {A,B}

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\, [\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \tag{35.28}$$

$$= \mathscr{P}_{pulse}(\omega)\, [\; p(1\text{-}p)(A\text{-}B)^2 + (pA + (1\text{-}p)B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)] \quad . \tag{35.35}$$

For the interesting case that p = 1/2 this becomes

$$\mathscr{P}(\omega) = \mathscr{P}_{\mathbf{pulse}}(\omega) \left[\frac{1}{4} (A\text{-}B)^2 + \frac{1}{4} (A\text{+}B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\right] \tag{35.36}$$

For the classic unipolar case A = 1 and B = 0, both coefficients are 1/4. For bipolar with A = 1 and B = -1 the coefficients are 1 and 0. Since the mean is then μ= 0, the discrete spectrum goes away. Since for both amplitudes $|y_m| = 1$, we are not surprised to find $\sigma^2 = 1$.

(i) Statistical Uncorrelated Pulse Trains: Summary and Example

Here then is a brief summary of the above results:

Spectral Power Density of an Uncorrelated Statistical Pulse Train (35.37)

$$x(t) = \sum_{n=-\infty}^{\infty} y_n \, x_{\mathbf{pulse}}(t - nT_1) \qquad // \text{ or } \sum_{n=-N}^{N} \text{ for a finite pulse train}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\right] \qquad // \text{ infinite} \qquad \textit{(35.28)}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \mu^2 \, 2\pi \, \delta_6(\omega T_1, N) \right] \qquad // \text{ finite} \qquad \textit{(35.17)}$$

If symbols are restricted to {A,B} then :

$$\sigma^2 = p(1\text{-}p)(A\text{-}B)^2 \qquad \text{and for } p = 1/2 \qquad \sigma^2 = \frac{1}{4}(A\text{-}B)^2 \qquad \textit{(35.34)}$$

$$\mu = pA + (1\text{-}p)B \qquad \text{and for } p = 1/2 \qquad \mu^2 = \frac{1}{4}(A\text{+}B)^2 \qquad \textit{(35.31)}$$

<u>Example</u>: Suppose A = 1 and B = 0 so that σ^2 = p(1-p) and $\mu^2 = p^2$. If we go on to assume p = 1, then every pulse has amplitude 1 and we have a simple pulse train. In this case σ^2 = 0 and μ^2 = 1 so

$$\mathscr{P}(\omega) \equiv \mathscr{P}_{\mathbf{pulse}}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad \text{joules}$$

which agrees with our infinite simple pulse train result (33.25). As we reduce p below p = 1, the line spectra are scaled down by $\mu^2 = p^2 < 1$, and a continuous spectrum starts to appear with σ^2 = p(1-p). The randomness (variance σ^2) of the amplitude magnitudes creates a continuous component in the spectral power density. As p reaches p = 1/2, we get σ^2 = 1/4 and μ^2 = 1/4 to give this classic result

$$\mathscr{P}(\omega) = \mathscr{P}_{\mathbf{pulse}}(\omega) \left[(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\right] \qquad p = 1/2 \tag{35.38}$$

Finally, when p reaches p = 0, then every pulse has B = 0, x(t) ≡ 0, σ^2 = 0, μ^2 = 0, and so $\mathscr{P}(\omega)$ = 0. There is nothing left.

(j) A numerical example of a Statistical Pulse Train

Recall from box (34.4) that for a finite pulse train,

$$\mathscr{P}(\omega) = \frac{|X(\omega)|^2}{2\pi T} = \frac{|X(\omega)|^2}{2\pi(2N+1)T_1} \qquad \mathscr{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} . \tag{35.39}$$

Therefore we can write the last line of (35.17) as,

$$\frac{<|X(\omega)|^2>}{(2N+1)} = |X_{pulse}(\omega)|^2 \, [(1/4) + (1/4)\, 2\pi\, \delta_6(\omega T_1,N)\,] \quad . \tag{35.40}$$

We shall use for $x_{pulse}(t)$ a square pulse of height A=1 and width $\tau = T_1 = 1$ so that, from (9.2),

$$|\, X_{pulse}\, (\omega)\, | = \text{sinc}(\omega/2) \; . \tag{35.41}$$

Since our pulse train will be fairly short (N = 20 pulses) and since we shall only average a small number of pulse trains (M = 10), we know our result will not exactly match (35.40). Still, we hope to see in our result some kind of continuous background spectrum which approximates the curve $(1/4)|X_{pulse}(\omega)|^2 = (1/4)\,\text{sinc}^2(\omega/2)$, and we expect to see delta-function-like peaks which, since $2\pi\delta_6(0,N) = (2N+1)$, have a peak value of about $(1/4)41 = 10.25$. Since this will be added to the continuous background, the central peak should have a height of $10.25 + .25 = 10.5$. However, for our small ensemble, we won't have exactly p = 1/2, so the delta peak won't be exactly 10.5 units high.

We know that δ_6 has identical peaks spaced by 2π, but we expect the non-central peaks to be suppressed by the $\text{sinc}^2(\omega/2)$ zeros which occur at $\omega = m(2\pi)$.

Here is a self-documented Maple program which generates $<|X(\omega)|^2>$. The program also generates the quantity $<X(\omega)>$ upon which we shall comment in section (k) below.

```
restart; assume(w,real); with(plots):
```
T1 is pulse duration, N is number of pulses in each pulse train
```
T1 := 1: N := 20:
```
Spectrum for standard box shaped pulse
```
Xpulse := T1*sin(w*T1/2)/(w*T1/2):
```
Subsequent calls to r() will generate either 0 or 1 with prob p = 1/2
```
r := rand(0..1):
```
Clear averaging accumulators. M is number of pulse trains to average

Xas means X absolute-value squared
```
X_acc := 0:Xas_acc := 0: M := 10:
for m from 1 to M do
```
Create a random pulse train
```
  for n from -N to N do y[n] := r(); od:
```
Compute the Digital Fourier Transform Y'(w) of the pulse train from (23.5)
```
  Yp := T1*sum(y[j]*exp(-I*w*j*T1),j=-N..N):
```
Compute X(w) for the pulse train from (25.4), omitting Xpulse(w)
```
  X := (1/T1)* Yp;
```
Accumulate both X(w) and |X(w)|^2, divide by M later for ensemble average
```
  X_acc := X_acc + X;
  Xas_acc := Xas_acc + eval(abs(X)^2);
od:
```
Compute <X(w)> and <|X(w)|^2> from the accumulators, but Xpulse not yet added
```
X_av_  :=  X_acc/M:
Xas_av_ := Xas_acc/M:
```

At this point, before $X_{\texttt{pulse}}(\omega)$ is added to the result, we plot $<|X(\omega)|^2>$. As expected, we see the peaks of δ_6 spaced by 2π and having height around 10 units,

Plot <|X(w)|^2>/(2N+1) of (35.40) without the pulse shape
```
plot(Xas_av_/(2*N+1), w=-10..10,color=[red,blue],view=0..12,numpoints=500);
```

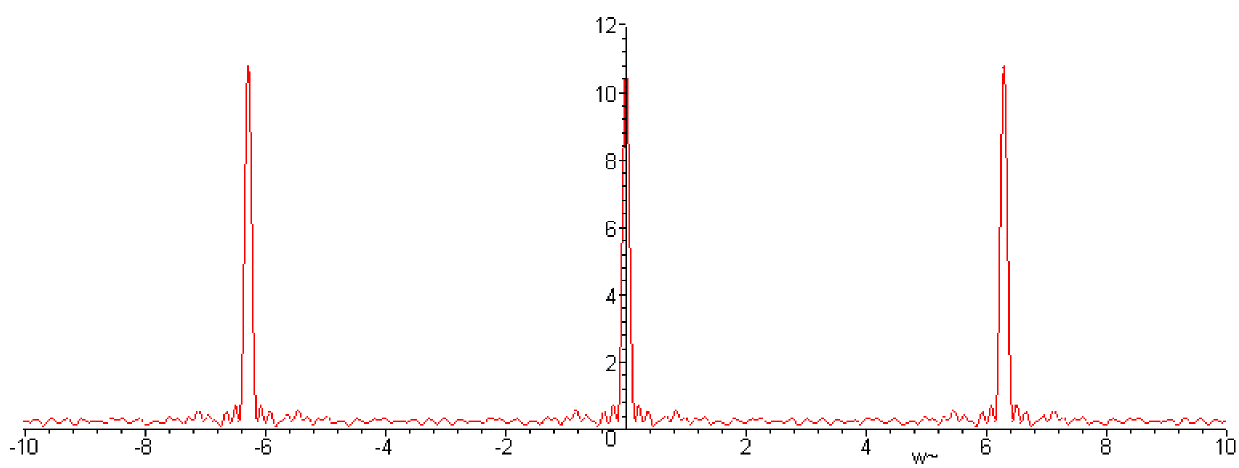


Fig 35.4

We now insert copies of $X_{\texttt{pulse}}(\omega)$ as appropriate,

Install the pulse shape to get the true expressions for <X(w)> and of <|X(w)|^2>

```
X_av := Xpulse*X_av_:
Xas_av := Xpulse^2*Xas_av_:
```

and then we can plot $\frac{<|X(\omega)|^2>}{(2N+1)}$ for ω in the same range (-10,10)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..12 ,color =
[red,blue],numpoints = 500);
```

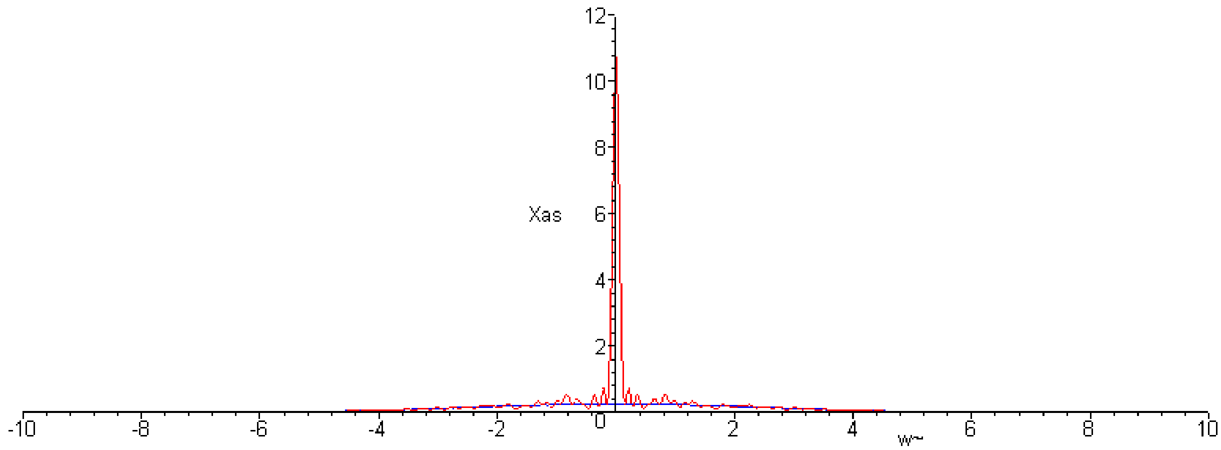


Fig 35.5

We see that the zeros of the sinc function have killed off the adjacent peaks.

Next, we restrict the plot height to be 0.8 units to view the detail, chopping off the central δ_6 peak,

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=-10..10, view=0..0.8 ,color =
[red,blue],numpoints = 500);
```

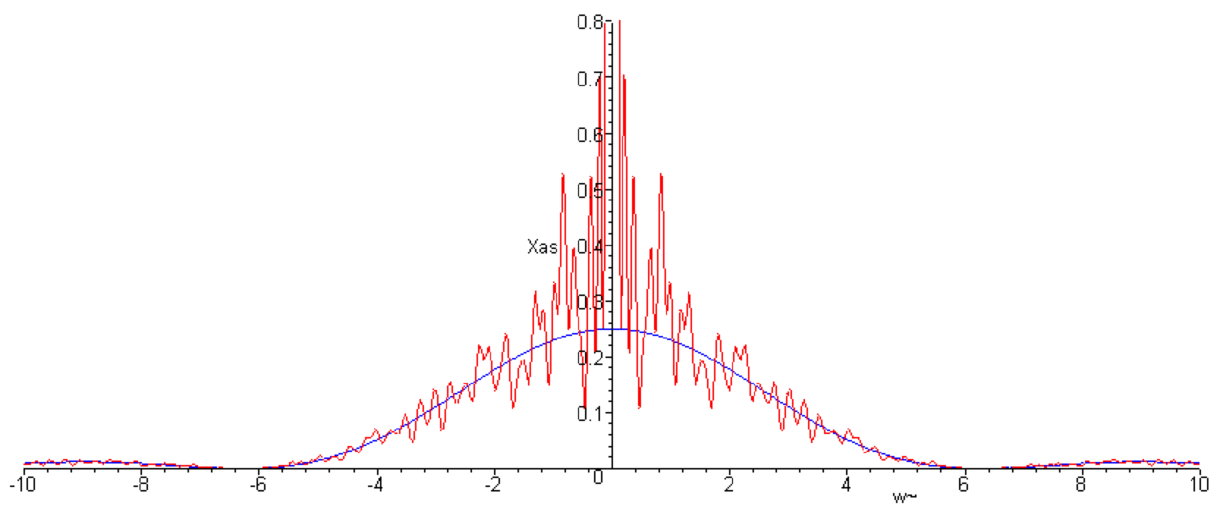


Fig 35.6

The spectrum $\frac{<|X(\omega)|^2>}{(2N+1)}$ is seen to have a continuous component which very well approximates one quarter of the sinc^2 curve, as we hoped it would. This tracking also occurs away from the central peak. Here is a blow-up of the above plot for ω in the range (5, 30)

```
plot([Xas_av/(2*N+1),(1/4)*Xpulse^2], w=5..30, view=0..0.02 ,color =
[red,blue],numpoints = 500);
```

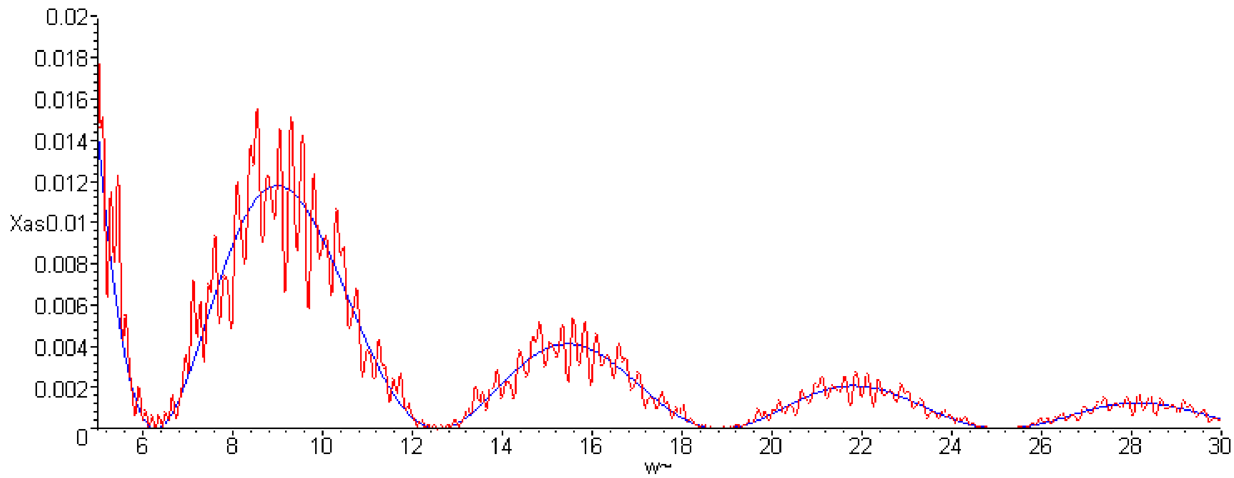


Fig 35.7

It might be noted that Maple does this work analytically, so that Xas-av is a function of ω having a large number of trigonometric terms. For the reader's interest, we show Xas-av(ω) for a typical run:

$$Xas\_av := 4\sin\left(\frac{1}{2}w\right)^2 \left(\frac{1}{10}(2\cos(20w)+2\cos(18w)+\cos(17w)+\cos(14w)+\cos(12w)+2\cos(11w)+2\cos(10w)\right.$$

$$+\cos(9w)+2\cos(8w)+2\cos(5w)+2\cos(3w)+\cos(2w)+\cos(w)+\cos(4w)+\cos(13w)+\cos(15w)+\cos(19w))^2$$

$$+\frac{1}{10}(\sin(17w)+\sin(14w)+\sin(12w)+\sin(9w)+\sin(2w)+\sin(w)-\sin(4w)-\sin(13w)-\sin(15w)-\sin(19w))^2+\frac{1}{10}(1$$

$$+2\cos(19w)+2\cos(16w)+2\cos(14w)+2\cos(13w)+2\cos(11w)+\cos(10w)+2\cos(8w)+2\cos(5w)+2\cos(4w)$$

$$+\cos(2w)+\cos(6w)+\cos(7w)+\cos(9w)+\cos(17w)+\cos(20w))^2$$

$$+\frac{1}{10}(\sin(10w)-\sin(2w)-\sin(6w)-\sin(7w)-\sin(9w)-\sin(17w)-\sin(20w))^2+\frac{1}{10}(1+\cos(19w)+2\cos(15w)$$

many terms omitted

$$+\cos(18w)+\cos(19w))^2+\frac{1}{10}$$

$$(\sin(15w)+\sin(12w)+\sin(w)-\sin(2w)-\sin(4w)-\sin(6w)-\sin(9w)-\sin(10w)-\sin(17w)-\sin(18w)-\sin(19w))^2+$$

$$\frac{1}{10}(2\cos(20w)+2\cos(19w)+2\cos(18w)+\cos(15w)+\cos(14w)+\cos(11w)+2\cos(9w)+2\cos(5w)+\cos(4w)$$

$$+2\cos(3w)+\cos(w)+\cos(12w)+\cos(16w)+\cos(17w))^2$$

$$\left.+\frac{1}{10}(\sin(15w)+\sin(14w)+\sin(11w)+\sin(4w)+\sin(w)-\sin(12w)-\sin(16w)-\sin(17w))^2\right) \Big/ w^2$$

In more serious work with larger numbers, one would of course do this in a more numeric fashion, but we are able to confirm the basic results even with this small experiment.

(k) A paradox and its resolution

Now while here, we can back up and apply our statistical average directly to the spectrum in (34.6) using (34.7). The result is

$$\langle X(\omega)\rangle = X_{pulse}(\omega) \sum_{n=-\infty}^{\infty} \langle y_n\rangle\ e^{-in\omega T_1} \quad . \qquad \textit{(34.6,7)} \qquad (35.42)$$

From (35.31) we have

$$\langle y_n\rangle = [p]\ A + [1-p]\ B \qquad = p \qquad \text{if } A=1, B=0 \quad . \qquad (35.43)$$

In this case, we can extract p from the above sum, which then collapses to form the usual (13.2) delta function sum. The result is then the same as the regular pulse train result (14.4) with an overall factor of p out front, namely,

$$\langle X(\omega)\rangle \ = p \sum_{m=-\infty}^{\infty} X_{pulse}(m\omega_1)\ 2\pi\ \delta(\ \omega T_1 - 2\pi m) \ . \qquad (35.44)$$

This says that our average spectrum is 100% *discrete*, there is no continuous part! If p = 1/2, the average spectrum is just 1/2 times our discrete unit amplitude pulse train spectrum (14.4). How can this be true, if we just showed in (35.28) that the average spectral density $\langle|X(\omega)|^2\rangle$ has a *continuous* spectral component?

The answer lies in the fact that $\langle ab\rangle \neq \langle a\rangle\langle b\rangle$, where $\langle\ \rangle$ is our averaging operation. Thus

$$\langle|X(\omega)|^2\rangle \ \neq\ \langle X(\omega)\rangle\langle X(\omega)^*\rangle \ . \qquad (35.45)$$

In particular we have from the second last line of (35.9) and (35.42),

$$\sum_{m=-N}^{N} \sum_{n=-N}^{N} \langle y_m\, y_n\rangle\ z^{m-n} \ \neq\ (\ \sum_{n=-N}^{N} \langle y_n\rangle\, z^{-n}\ )\,(\ \sum_{m=-N}^{N} \langle y_m\rangle\, z^{m}\ )\ . \qquad (35.46)$$

These double sums *would* be equal if $\langle y_m\, y_n\rangle = \langle y_n\rangle\langle y_m\rangle$ for all m and n, but this is not true when n = m. The left sum in that case sees $\langle y_m^2\rangle = \sigma^2+\mu^2$ while the right side sees $\langle y_m\rangle^2 = \mu^2$.

In general we do not expect the average of a product to be the product of the averages,

$$(\frac{1}{N}\Sigma_{i=1}^{N} A_iB_i) \neq (\frac{1}{N}\Sigma_{i=1}^{N} A_i)\,(\frac{1}{N}\Sigma_{i=1}^{N} B_i) \ .$$

In the numerical example presented in the previous section we computed $\langle X(\omega)\rangle$ for a small ensemble of pulse trains. Here are plots of the real and imaginary part of $\langle X(\omega)\rangle$,

```
plot([Re(X_av),Im(X_av)], w=-10..10, view =-6..22 ,color = [red,blue],numpoints = 500);
```

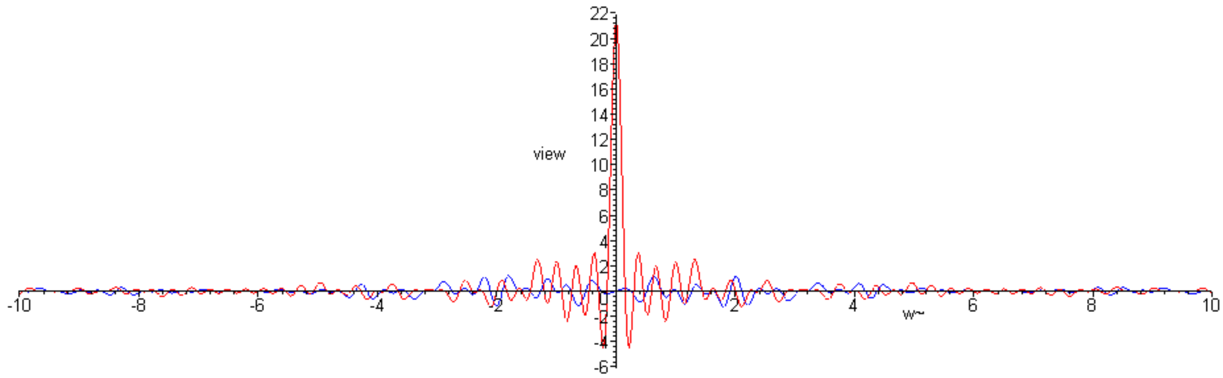


Fig 35.8

We can see that, apart from the noise of our low statistics, there is only the central peak and no continuous spectrum component. A blow-up of the central region follows,

```
plot([Re(X_av),Im(X_av)], w=-4..4, view =-6..8 ,color = [red,blue],numpoints = 500);
```

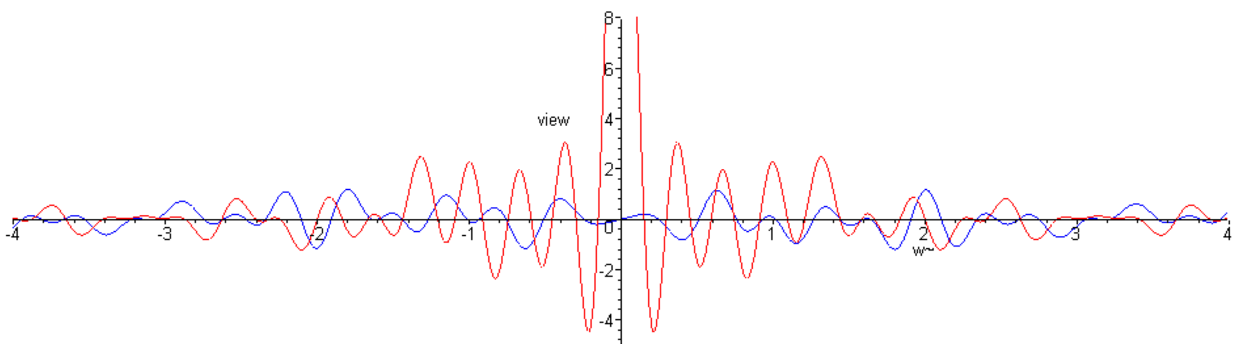


Fig 35.9

(ℓ) What role has the Autocorrelation Function played in our development?

In **Section 32** the autocorrelation function was first introduced as $r_x(t)$ for x(t), and was computed for the simple case of a box pulse. Treating the definition of $r_x(t)$ as a convolution equation, the Wiener-Khintchine relation $R_x(\omega) = |X(\omega)|^2$ was trivially derived, where $R_x(\omega)$ is the Fourier Integral Transform of $r_x(t)$. It was then shown that the spectral energy density of a pulse train is $\mathcal{E}(\omega) = (1/2\pi)\, R_x(\omega)$ due to this relation. It was then shown that $r_x(0) = E$, the total energy in the pulse train. We then commented on the origin of the name, showing that the auto-correlation function is the cross-correlation function of a function with itself. Finally, we developed the Z Transform version of the Wiener-Khintchine Relation $R''(z) = (T_1/T)\,|\, Y''(z)\,|^2$ where R"(z) is the Z Transform of the autocorrelation sequence r_s.

Section 33 (simple pulse trains) made no mention of autocorrelation.

In **Section 34** (general pulse trains) it was noted again that $P = r_x(0)/T$ since $P = E/T$, and that $\mathcal{P}(\omega) = R_x(\omega)/\ (2\pi T)$ since $R_x(\omega) = |X(\omega)|^2$. At the end of section (b) both $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$ are stated in terms of R"(z) . In particular, (34.14a) says $\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$ where $R''(z) = \Sigma_s\, r_s z^{-s}$, and this expression for $\mathcal{P}(\omega)$ appears throughout the current Section 35.

<u>Comments</u>:

(1) In the Two Methods discussion above, we showed that the autocorrelation method is only one of two approaches one might take to obtain $\mathcal{P}(\omega)$. Thus, it is always possible to compute $\mathcal{P}(\omega)$ without ever using or even knowing about the autocorrelation sequence r_s. For example, in (35.17) the autocorrelation method leads to a recalcitrant sum evaluation (Appendix D), whereas the double-sum method leads to an easier discovery of $\mathcal{P}(\omega)$, as shown directly in (35.19).

(2) In some textbooks, one gets the impression that the autocorrelation function is somehow indispensable for the development of the $\mathcal{P}(\omega)$ equations. It is not, but it is convenient for many applications. We will use the autocorrelation method several times in the remaining Sections.

36. Application to some Standard Non-Correlated Pulse Train Types (Line Codes)

Here we apply our boxed results (35.37) to statistical pulse trains of various types. When a pulse train is in fact a voltage on a pair of wires (transmission line, such as a telephone "line"), the way in which signals are encoded in the pulse train is called a **line code**. One could consider a random speed Morse code signal going down a wire as a line code, but the term usually refers to a sequence of equally spaced amplitude modulated pulses, meaning a pulse train. Often line codes get modulated onto an RF carrier, in which case the line code is thought of as the baseband signal prior to modulation. For this reason, line codes are often discussed in the "baseband chapter" of any digital communications text.

The line code names are a little strange due to their history. Here are the pulse shapes used for RZ and NRZ lines codes. In either case a 1 is (is coded as) a pulse and a 0 is no pulse.

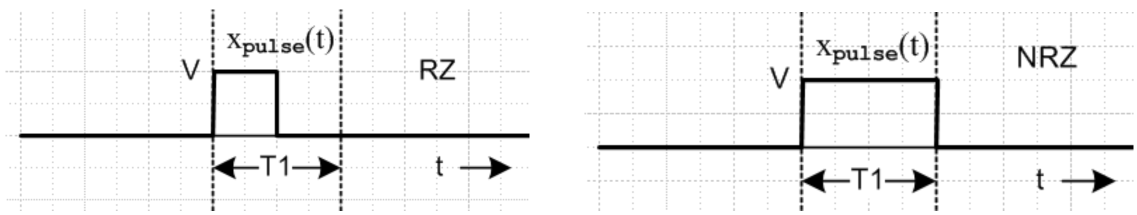


Fig 36.1

On the left, since the signal returns to zero inside the pulse period, it is called a "return to zero" code RZ. Since this does not happen on the right, that is a "non return to zero" code, NRZ.

(a) Unipolar NRZ line code

Pulse Shape. The pulse is a box of amplitude V and width $\tau = T_1$,

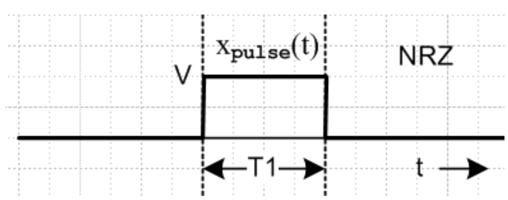


Fig 36.2

From (9.2) we know that

$$X_{pulse}(\omega) = (VT_1)\ \mathrm{sinc}(\omega T_1/2) = (VT_1)\ \mathrm{sinc}(\pi \frac{\omega}{\omega_1}) \qquad \omega_1 = 2\pi/T_1$$

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (VT_1)^2\ \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1})/(2\pi T_1) = (V^2/\omega_1)\ \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \qquad (36.1)$$

Coding: NRZ is a normal binary signal, high for period T_1 to indicate a 1, and low for T_1 to indicate a 0. Sometimes this is called unipolar NRZ since the signal never goes negative.

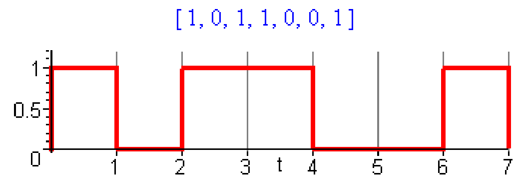


Fig 36.3

Coefficients σ^2 and μ^2 : From summary box (35.37),

$$A = 1 \text{ and } B = 0 \Rightarrow \qquad \sigma^2 = p(1-p) \qquad \mu^2 = p^2 \qquad \mu = p$$

Spectrum: The average spectral power density for the NRZ line code is :

$$\mathcal{P}(\omega) = \sigma^2 \mathcal{P}_{pulse}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{pulse}(m\omega_1)\, \delta(\frac{\omega}{\omega_1} - m) \qquad (35.28d)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [\, p(1-p)\, \text{sinc}^2(\pi\frac{\omega}{\omega_1}) + p^2 \sum_{m=-\infty}^{\infty} \text{sinc}^2(\pi\frac{\omega}{\omega_1})\, \delta(\frac{\omega}{\omega_1} - m)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [\, p(1-p)\, \text{sinc}^2(\pi\frac{\omega}{\omega_1}) + p^2\, \delta(\frac{\omega}{\omega_1})\,] \qquad \text{// any p} \qquad (36.2)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [\, \frac{1}{4}\, \text{sinc}^2(\pi\frac{\omega}{\omega_1}) + \frac{1}{4}\, \delta(\frac{\omega}{\omega_1})\,] \qquad \text{// for p = 1/2} \qquad (36.3)$$

The sinc function killed off all but the m = 0 line (the "DC line").

In order to express this (and later) results in the frequency domain, we use these relations

$$\mathcal{P}(\omega) = P(f)/2\pi \qquad \text{// (34.4) bottom line}$$
$$1/\omega_1 = T_1/2\pi$$
$$\frac{\omega}{\omega_1} = \frac{f}{f_1} \equiv x \text{ for plots} \qquad \text{// } f_1 = 1/T_1 \qquad (36.4)$$

to obtain

$$P(f) = V^2 T_1\, [\, p(1-p)\, \text{sinc}^2(\pi\frac{f}{f_1}) + p^2\, \delta(\frac{f}{f_1})\,] \qquad \text{// any p} \qquad (36.2)'$$

$$P(f) = V^2 T_1\, [\, \frac{1}{4}\, \text{sinc}^2(\pi\frac{f}{f_1}) + \frac{1}{4}\, \delta(\frac{f}{f_1})\,] \qquad \text{// for p = 1/2} \qquad (36.3)'$$

This last line can be written

$$P(f) = V^2\, [\, \frac{1}{4}\, T_1\, \text{sinc}^2(\pi f T_1) + \frac{1}{4}\, \delta(f)\,] \qquad \text{// for p = 1/2} \qquad (36.3)''$$

which then agrees with Xiong (2.25).

Plot: Ignoring the overall factor (V^2/ω_1) we make this plot of $\mathscr{P}(\omega)$ given by (36.2):

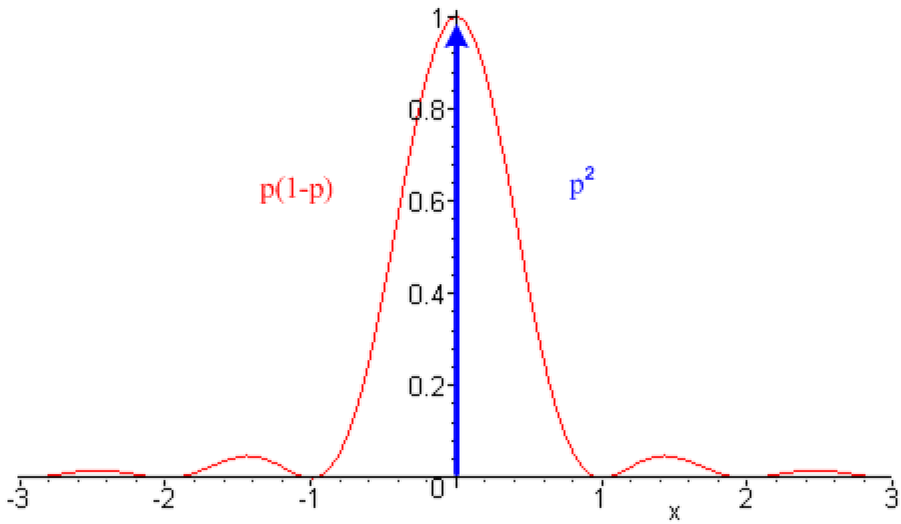


Fig 36.4

The red curve should be scaled by the red factor shown on the left, and the blue delta line should be scaled by the blue factor on the right.

Power Partition: The total power in the continuous part of the spectrum is: ($d\omega = \omega_1 dx$)

$$\text{AC power} = \int_{-\infty}^{\infty} \omega_1 dx\, \mathscr{P}(x\omega_1> = p(1-p)\, V^2 \int_{-\infty}^{\infty} dx\, \text{sinc}^2(\pi x) = p(1-p)\, V^2 \ .$$

The total power in the DC line at $\omega = 0$ is

$$\text{DC power} = \int_{-\infty}^{\infty} \omega_1 dx\, \mathscr{P}(x\omega_1) = V^2 \int_{-\infty}^{\infty} dx\, p^2\, \delta(x) = p^2 V^2$$

Thus we find that

$$\text{total power} = \underbrace{p^2\, V^2}_{\text{DC}} + \underbrace{p(1-p)\, V^2}_{\text{AC}} = pV^2$$

If p = 1/2, then

$$\text{total power} = \underbrace{(1/4)\, V^2}_{\text{DC}} + \underbrace{(1/4)\, V^2}_{\text{AC}} = (1/2)V^2 \qquad (36.5)$$

so half the power is in the DC line and half in the AC signal. The DC term is certainly reasonable since we know that with p = 1/2, the average voltage is (V/2).

If one wanted to reduce wasted power, it would be good to give this signal a DC offset of -V/2 and then there would be no DC line. This is in fact the next example if one takes $V \rightarrow V/2$.

(b) Bipolar NRZ line code

Pulse Shape. The pulse shape is the same as for Unipolar NRZ

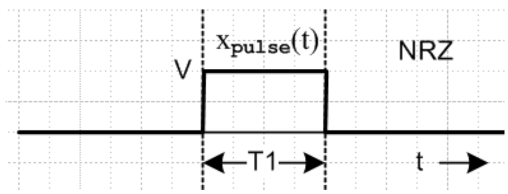


Fig 36.2

$$\mathcal{P}_{pulse}(\omega) = (V^2/\omega_1)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \quad \text{same as for unipolar NRZ} \qquad (36.1)$$

Coding: 1 is coded as a positive box with amplitude V, and a 0 as a negative box having amplitude -V.

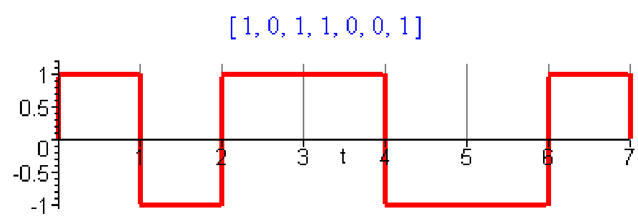


Fig 36.5

Coefficients σ^2 and μ^2 : From summary box (35.37),

A = 1 and B = -1 => $\sigma^2 = 4p(1-p)$ $\quad \mu^2 = (1-2p)^2$ $\quad \mu = 2p-1$

Spectrum: The average spectral power density for the bipolar NRZ line code is :

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{pulse}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{pulse}(m\omega_1)\, \delta(\frac{\omega}{\omega_1} - m) \qquad (35.28d)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [4p(1-p)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) + (1-2p)^2 \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1})\, \delta(\frac{\omega}{\omega_1} - m)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [4p(1-p)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) + (1-2p)^2\, \delta(\frac{\omega}{\omega_1})\,] \qquad \text{// any p} \qquad (36.6)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1})\,] \qquad \{ = \mathcal{P}_{pulse}(\omega) \} \qquad \text{// for p = 1/2} \qquad (36.7)$$

which become, using (36.4),

$$P(f) = (V^2 T_1)\, [4p(1-p)\, \mathrm{sinc}^2(\pi \frac{f}{f_1}) + (1-2p)^2\, \delta(\frac{f}{f_1})\,] \qquad \text{// any p} \qquad (36.6)'$$

$$P(f) = (V^2 T_1)\, [\, \mathrm{sinc}^2(\pi \frac{f}{f_1})\,] \qquad \{ = P_{pulse}(f) \} \qquad \text{// for p = 1/2} \qquad (36.7)'$$

This last result agrees with Xiong (2.20). The spectrum is all continuous when p = 1/2 since then the DC portion is killed off. A discrete spectrum cannot exist if the waveform amplitudes have zero mean, $\mu = 0$.

Notice that for p = 1/2, bipolar NRZ has $\mathscr{P}(\omega) = \mathscr{P}_{\text{pulse}}(\omega)$, so the statistical pulse train spectrum is the same as that of the underlying pulse.

Plot: Ignoring the overall factor (V^2/ω_1) we make this plot of $<\mathscr{P}(x\omega_1)>$ given by (36.6)

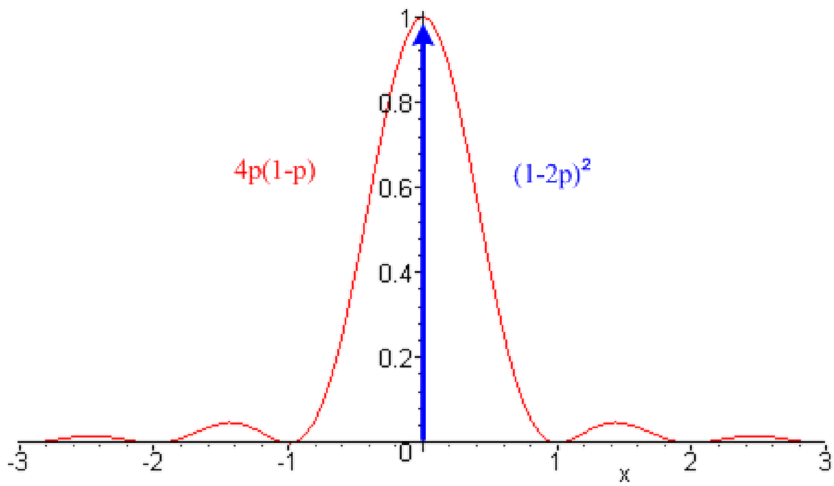


Fig 36.6

which is the same as the spectrum for unipolar NRZ except for the two scaling factors.

Power Partition: We can again compute the DC and AC power.

AC power = unipolar NRZ with $p(1-p) \rightarrow 4p(1-p)$, so AC = $4p(1-p)\,V^2$

DC power = unipolar NRZ with $p^2 \rightarrow (2p-1)^2$, so DC = $(2p-1)^2\,V^2$

$$\text{total power} = \underbrace{(2p-1)^2V^2}_{DC} + \underbrace{4p(1-p)V^2}_{AC} = V^2 \quad \text{, independent of p.} \tag{36.8}$$

The total power is independent of p because a pulse has the same AC power if it goes up or down. For p = 1/2 we get

$$\text{total power} = \underbrace{0}_{DC} + \underbrace{V^2}_{AC} = V^2 \qquad // \; p = 1/2$$

and now no power is wasted pushing DC through a line. If we take V→V/2 to have a comparable peak-to-peak amplitude, we find

AC power = $(V/2)^2$

which is the same as the AC power in (36.5); it is not affected by a DC offset of -V/2.

(c) Unipolar RZ line code

Pulse Shape. Here the basic pulse is a box that fills only half the time interval T_1.

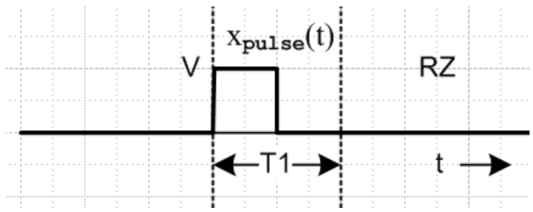


Fig 36.7

According to (12.1) applied to the above pulse.

$$\begin{aligned} x(t) &\leftrightarrow X(\omega) \\ x(t - T_1/2) &\leftrightarrow X(\omega)\, e^{-i\omega T_1/2} \quad . \end{aligned} \qquad (12.1)$$

where $X(\omega)$ is for a pulse of total width $T_1/2$ *centered* at $t = 0$. The spectrum of this centered pulse is given by (9.2) with $\tau = T_1/2$ as $X(\omega) = (VT_1/2)\,\text{sinc}(\omega T_1/4)$. Thus, the above pulse has this spectrum

$$X_{pulse}(\omega) = e^{-i\omega T_1/2}\,(VT_1/2)\,\text{sinc}(\omega T_1/4) = e^{-i\omega T_1/2}\,(VT_1/2)\,\text{sinc}\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)$$

and then

$$\mathscr{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = \frac{(VT_1/2)^2}{2\pi T_1}\,\text{sinc}^2\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right) = (V^2/4\omega_1)\,\text{sinc}^2\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right) . \qquad (36.9)$$

Coding: 1 is coded as the presence of the pulse, 0 is coded as the absence of a pulse.

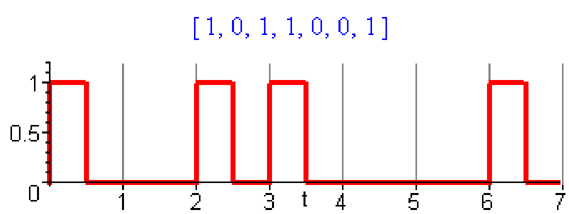


Fig 36.8

Coefficients σ^2 and μ^2 : From summary box (35.37), coefficients are the same as for NRZ,

A = 1 and B = 0 => $\sigma^2 = p(1\text{-}p)$ $\mu^2 = p^2$ $\mu = p$

Spectrum: The average spectral power density for the unipolar RZ line code is :

$$\mathscr{P}(\omega) = \sigma^2\,\mathscr{P}_{pulse}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathscr{P}_{pulse}(m\omega_1)\,\delta\left(\frac{\omega}{\omega_1} - m\right) \qquad (35.28d)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\,[\,p(1\text{-}p)\,\text{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + p^2 \sum_{m=-\infty}^{\infty} \text{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\,\delta(\frac{\omega}{\omega_1} - m)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\,[\,p(1\text{-}p)\,\text{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + p^2 \sum_{m = \text{odd}} \text{sinc}^2(\frac{\pi}{2}m)\,\delta(\frac{\omega}{\omega_1} - m) + p^2\,\delta(\frac{\omega}{\omega_1})\,] \qquad (36.10)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\,[\,\frac{1}{4}\,\text{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + \frac{1}{4} \sum_{m = \text{odd}} \text{sinc}^2(\frac{\pi}{2}m)\,\delta(\frac{\omega}{\omega_1} - m) + \frac{1}{4}\,\delta(\frac{\omega}{\omega_1})\,] \qquad (36.11)$$

where the last line is for $p = 1/2$. Then using (36.4) we write the f domain versions,

$$P(f) = (V^2T_1/4)\,[\,p(1\text{-}p)\,\text{sinc}^2(\frac{\pi}{2}\frac{f}{f_1}) + p^2 \sum_{m = \text{odd}} \text{sinc}^2(\frac{\pi}{2}m)\,\delta(\frac{f}{f_1} - m) + p^2\,\delta(\frac{f}{f_1})\,] \qquad (36.10)'$$

$$P(f) = (V^2T_1/4)\,[\,\frac{1}{4}\,\text{sinc}^2(\frac{\pi}{2}\frac{f}{f_1}) + \frac{1}{4} \sum_{m = \text{odd}} \text{sinc}^2(\frac{\pi}{2}m)\,\delta(\frac{f}{f_1} - m) + \frac{1}{4}\,\delta(\frac{f}{f_1})\,] \qquad (36.11)'$$

In all these expressions one can replace $\text{sinc}^2(\frac{\pi}{2}m)$ by its odd-integer value $\frac{4}{\pi^2}\frac{1}{m^2}$ since

$$\text{sinc}^2(\frac{\pi}{2}\,m) = \begin{cases} 1 & m = 0 \\ 0 & m = \text{even} \neq 0 \\ \frac{4}{\pi^2}\frac{1}{m^2} & m = \text{odd} \end{cases}$$

Result (36.11)' agrees with Xiong (2.31) in which $R_b \equiv 1/T$ = our $1/T_1$, but he has not separated out the three terms $m = 0$, $m = \text{even} \neq 0$ and $m = \text{odd}$.

Plot: Ignoring now the overall factor $(V^2/4\omega_1)$ we get this power spectrum $\mathcal{P}(\omega)$ from (36.10),

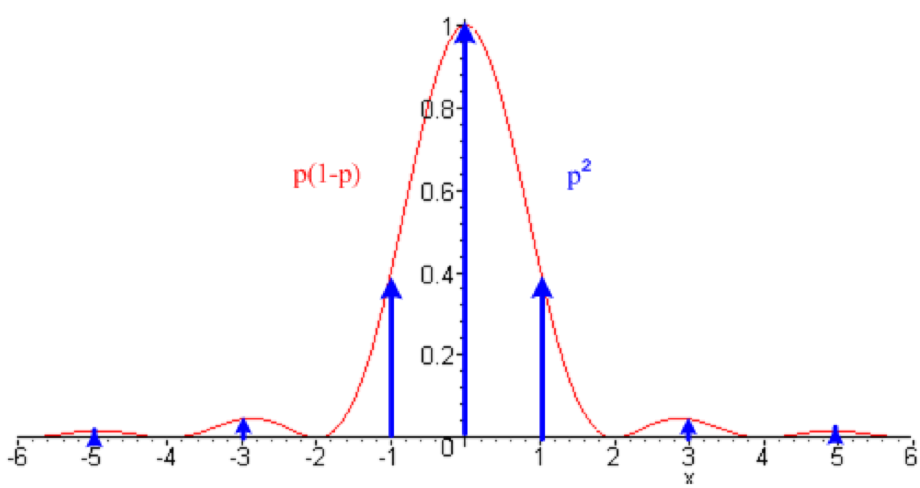


Fig 36.9

We now have three pieces: a continuous part, the DC line, and a set of lines at odd m. The main peak is twice as wide as the NRZ peak since the underlying pulse is half as wide.

Power Partition: Once again, we can compute the total power for each of these three pieces.

$$\text{odd lines power} = \int_{-\infty}^{\infty} \omega_1 dx\, \mathscr{P}(x\omega_1) = \omega_1 (V/2)^2 (1/\omega_1)\, p^2 \int_{-\infty}^{\infty} dx\, \text{sinc}^2(\frac{\pi}{2}x)\, 2 \sum_{m=1\,3\,5...}^{\infty} \delta(x-m)$$

$$= (V/2)^2\, 2p^2 \sum_{m=1\,3\,5...}^{\infty} \text{sinc}^2(\frac{\pi}{2}m) = (V/2)^2\, 2p^2 \sum_{m=1\,3\,5...}^{\infty} \frac{\sin^2(\frac{\pi}{2}m)}{(\frac{\pi}{2}m)^2} = (V/2)^2\, 2p^2\, (2/\pi)^2 \sum_{m=1\,3\,5...}^{\infty} \frac{1}{m^2}$$

$$= (V/2)^2\, 2p^2\, (2/\pi)^2\, (\pi^2/8) = p^2(V/2)^2$$

where the sum $\Sigma_{odd}(1/m^2) = \pi^2/8$ is from Gradshteyn and Ryzhik 0.234.2. Then

$$\text{DC power} = \int_{-\infty}^{\infty} \omega_1 dx\, \mathscr{P}(x\omega_1) = (V/2)^2 p^2 \int_{-\infty}^{\infty} dx\, \text{sinc}^2(\frac{\pi}{2}x)\, \delta(x) = p^2\, (V/2)^2$$

which is the same as the odd lines power. Finally,

$$\text{continuum power} = \int_{-\infty}^{\infty} \omega_1 dx\, \mathscr{P}(x\omega_1) = (V/2)^2\, p(1-p) \int_{-\infty}^{\infty} dx\, \text{sinc}^2(\frac{\pi}{2}x) = 2p(1-p)\, (V/2)^2$$

So the power partitioning is

$$\text{total power} = \underbrace{p^2(V/2)^2}_{\text{DC}} + \underbrace{p^2\,(V/2)^2}_{\text{other lines}} + \underbrace{2p(1-p)\,(V/2)^2}_{\text{continuum}}$$

$$= \underbrace{p^2(V/2)^2}_{\text{DC}} + \underbrace{p(2-p)\,(V/2)^2}_{\text{AC}} = 2p(V/2)^2 = (p/2)V^2 \,. \qquad (36.12)$$

This is half of the total power of unipolar NRZ (36.4), which seems reasonable since the pulses here have half the duration.

(d) Bipolar RZ line code

Pulse Shape. The pulse shape is the same as for Unipolar RZ

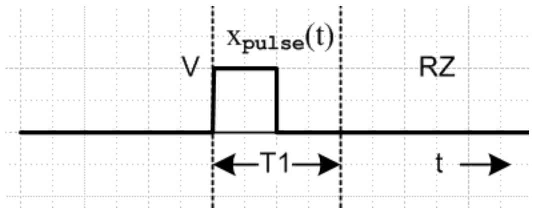


Fig 36.7

$$\mathcal{P}_{pulse}(\omega) = (V^2/4\omega_1)\ \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\ . \qquad (36.9)$$

Coding: 1 is coded as a positive pulse with amplitude V, and a 0 as a negative pulse having amplitude -V.

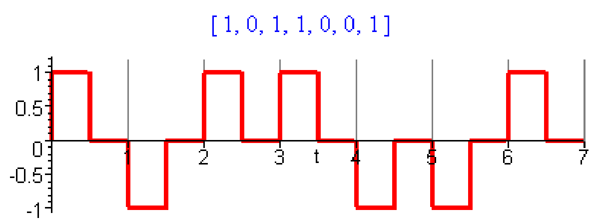


Fig 36.10

Coefficients σ^2 and μ^2 : From summary box (35.37), and the same as for bipolar NRZ

A = 1 and B = -1 => $\sigma^2 = 4p(1\text{-}p)$ $\mu^2 = (1\text{-}2p)^2$ $\mu = 2p\text{-}1$

Spectrum: The average spectral power density for the bipolar RZ line code is :

$$\mathcal{P}(\omega) = \sigma^2\, \mathcal{P}_{pulse}(\omega) + \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{pulse}(m\omega_1)\, \delta(\frac{\omega}{\omega_1} - m) \qquad (35.28d)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\, [4p(1\text{-}p)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + (1\text{-}2p)^2 \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\, \delta(\frac{\omega}{\omega_1} - m)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\, [4p(1\text{-}p)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + (1\text{-}2p)^2 \sum_{m=odd} \mathrm{sinc}^2(\frac{\pi}{2}\, m\,)\delta(\frac{\omega}{\omega_1} - m) + (1\text{-}2p)^2\, \delta(\frac{\omega}{\omega_1})] \quad (36.13)$$

$$\mathcal{P}(\omega) = (V^2/4\omega_1)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) \qquad \{ = \mathcal{P}_{pulse}(\omega) \} \qquad // \ p = 1/2 \qquad (36.14)$$

which become, using (36.4),

$$P(f) = (V^2T_1/4)\, [4p(1\text{-}p)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{f}{f_1}) + (1\text{-}2p)^2 \sum_{m=odd} \mathrm{sinc}^2(\frac{\pi}{2}\, m\,)\delta(\frac{f}{f_1} - m) + (1\text{-}2p)^2\, \delta(\frac{f}{f_1})\,] \quad (36.13)'$$

$$P(f) = (V^2T_1/4)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{f}{f_1}) \qquad \{ = P_{pulse}(f) \} \qquad // \ p = 1/2 \qquad (36.14)'$$

This last result agrees with Xiong (2.30). As noted earlier, $\text{sinc}^2(\frac{\pi}{2}m) = \frac{4}{\pi^2}\frac{1}{m^2}$ for odd m.

Plot: Ignoring now the overall factor $(V^2/4\omega_1)$ we get this power spectrum from (36.13),

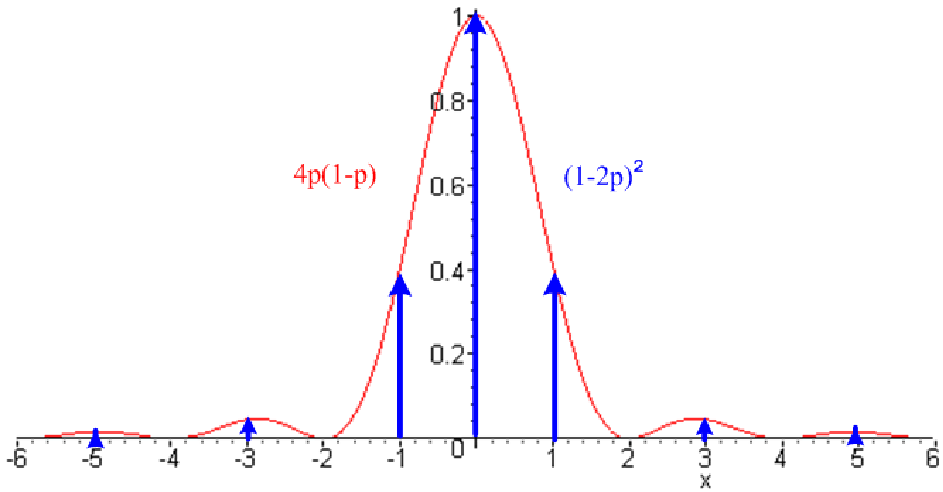


Fig 36.11

Power Partition: Once again, we can compute the total power for each of three pieces. These are the same as for the unipolar RZ if we make the replacements $p(1-p) \rightarrow 4p(1-p)$ and $p^2 \rightarrow (1-2p)^2$, so

$$\text{odd lines power} = (1-2p)^2\,(V/2)^2$$

$$\text{DC power} = (1-2p)^2\,(V/2)^2$$

$$\text{continuum power} = 8p(1-p)\,(V/2)^2$$

So the power partitioning is

$$\text{total power} = \underbrace{(1-2p)^2\,(V/2)^2}_{\text{DC}} + \underbrace{(1-2p)^2\,(V/2)^2}_{\text{odd lines}} + \underbrace{8p(1-p)\,(V/2)^2}_{\text{continuum}}$$

$$= \underbrace{(1-2p)^2\,(V/2)^2}_{\text{DC}} + \underbrace{[1-4p(p-1)]\,(V/2)^2}_{\text{AC}} = 2\,(V/2)^2 = V^2/2 \qquad (36.15)$$

As expected, this is half the total power of bipolar NRZ since the pulses have half the duration. For p = 1/2 all the power is in the continuum.

(e) Manchester line code

This line code was developed at the University of Manchester probably in the World War II era. At that time Tom Kilburn, Alan Turing and others were building the world's first stored-program computer.

Pulse Shape: The pulse shape here is the biphase (biphasic, diphase) pulse,

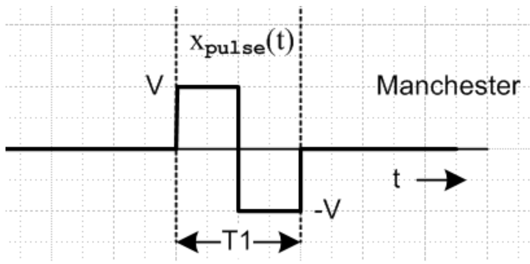


Fig 36.12

We already computed $X_{pulse}(\omega)$ for this pulse in (19.2), so we now set $\tau = T_1/2$ and $A/2 = V$ to get

$$X_{pulse}(\omega) = (4iV/\omega)\sin^2(\omega T_1/4) = (4iV/\omega)\sin(\omega T_1/4)\,[\sin(\omega T_1/4) / (\omega T_1/4)]\,(\omega T_1/4)$$

$$= (iVT_1)\sin(\omega T_1/4)\,\mathrm{sinc}(\omega T_1/4)$$

$$\mathscr{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (V^2/\omega_1)\sin^2\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)\mathrm{sinc}^2\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right) . \qquad (36.16)$$

This spectral pulse density is $4\sin^2\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)$times that of the RZ pulse shown in (36.9). This extra factor kills off the spectrum at $\omega = 0$.

Coding: 1 is coded as the above pulse, 0 is coded as the negative of the pulse (but some sources use the opposite polarity),

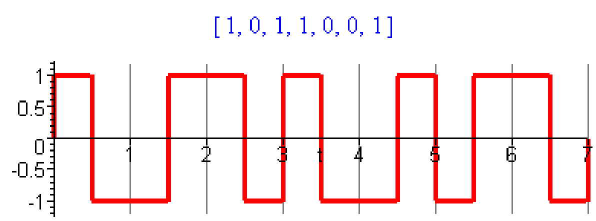


Fig 36.13

Coefficients σ^2 and μ^2 : From summary box (35.37), and the same as for bipolar NRZ

$A = 1$ and $B = -1$ => $\sigma^2 = 4p(1-p)$ $\mu^2 = (1-2p)^2$ $\mu = 2p-1$

Comment: Notice that the mean value of the waveform in Fig 36.13 is 0 regardless of p, whereas the mean value μ of the amplitudes y_n is given by $\mu = 2p-1$.

Spectrum: The average spectral power density for the Manchester code is :

$$\mathcal{P}(\omega) = \sigma^2 \mathcal{P}_{\mathbf{pulse}}(\omega) \;+\; \mu^2 \sum_{m=-\infty}^{\infty} \mathcal{P}_{\mathbf{pulse}}(m\omega_1)\, \delta(\frac{\omega}{\omega_1} - m) \qquad \textit{(35.28d)}$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [4p(1-p) \sin^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + (1-2p)^2 \sum_{m=-\infty}^{\infty} \sin^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\, \delta(\frac{\omega}{\omega_1} - m)]$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, [4p(1-p) \sin^2(\frac{\pi}{2}\frac{\omega}{\omega_1})\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) + (1-2p)^2 \sum_{m=\mathrm{odd}} \mathrm{sinc}^2(\frac{\pi}{2} m)\, \delta(\frac{\omega}{\omega_1} - m)\,] \qquad (36.17)$$

$$\mathcal{P}(\omega) = (V^2/\omega_1)\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{\omega}{\omega_1}) \qquad \{ = \mathcal{P}_{\mathbf{pulse}}(\omega) \} \qquad // \; p = 1/2 \qquad (36.18)$$

which become, using (36.4),

$$P(f) = (V^2T_1)\, [4p(1-p) \sin^2(\frac{\pi}{2}\frac{f}{f_1})\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{f}{f_1}) + (1-2p)^2 \sum_{m=\mathrm{odd}} \mathrm{sinc}^2(\frac{\pi}{2} m)\, \delta(\frac{f}{f_1} - m)\,] \qquad (36.17)'$$

$$P(f) = (V^2T_1) \sin^2(\frac{\pi}{2}\frac{f}{f_1})\, \mathrm{sinc}^2(\frac{\pi}{2}\frac{f}{f_1})] \qquad // \; p = 1/2 \qquad (36.18)'$$

As noted earlier, $\mathrm{sinc}^2(\frac{\pi}{2} m) = \frac{4}{\pi^2}\frac{1}{m^2}$ for odd m. The last result agrees with Xiong (2.38). He refers to this Manchester code as Bi-Φ-L.

Plot: Ignoring the leading factor (V^2/ω_1) the spectrum for (36.17) has this plot,

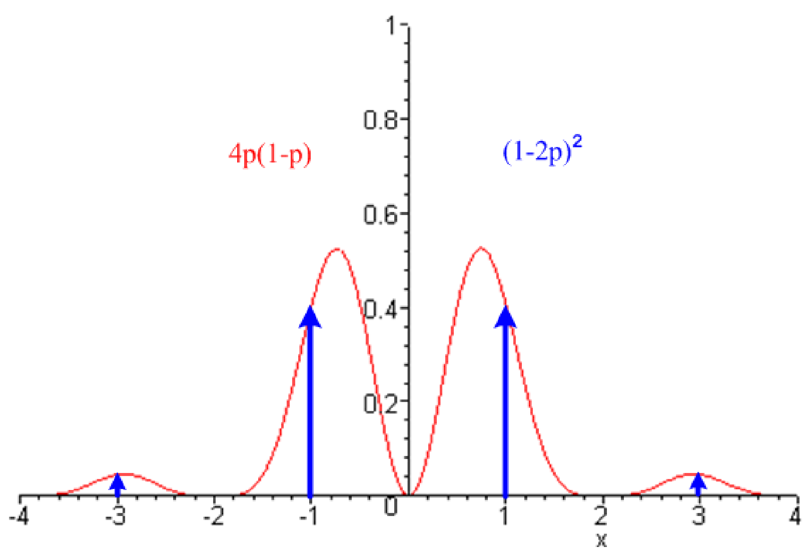


Fig 36.14

Power Partition:

$$\text{lines power} = \int_{-\infty}^{\infty} \omega_1 dx \langle\mathcal{P}(x\omega_1)\rangle = V^2\, (2p-1)^2 \int_{-\infty}^{\infty} dx \sin^2(\frac{\pi}{2} x)\, \mathrm{sinc}^2(\frac{\pi}{2} x) \sum_{m=\pm\mathrm{odd}} \delta(x - m)$$

$$= V^2 (2p-1)^2 \sum_{m = \pm odd} \sin^2(\tfrac{\pi}{2} m)\, \mathrm{sinc}^2(\tfrac{\pi}{2} m) = V^2 (2p-1)^2 \sum_{m = \pm odd} \sin^4(\tfrac{\pi}{2} m) (\tfrac{\pi}{2} m)^{-2}$$

$$= V^2 (2p-1)^2 (2/\pi)^2 \sum_{m = \pm odd} 1/m^2 = V^2 (2p-1)^2 (2/\pi)^2\, 2 \sum_{m = 1\,3\,5...}^{\infty} 1/m^2$$

$$= V^2 (2p-1)^2 (2/\pi)^2\, 2\, (\pi^2/8) = (2p-1)^2 V^2$$

$$\text{continuum power} = \int_{-\infty}^{\infty} \omega_1 dx \langle \mathscr{P}(x\omega_1) \rangle = V^2\, 4p(1-p) \int_{-\infty}^{\infty} dx\, \sin^2(\tfrac{\pi}{2} x)\, \mathrm{sinc}^2(\tfrac{\pi}{2} x) = V^2\, 4p(1-p)$$

since the integral is just 1. Therefore,

$$\text{total power} = \underset{DC}{0} + \underset{lines}{(2p-1)^2 V^2} + \underset{continuum}{4p(1-p) V^2} = \underset{AC}{V^2} \qquad (36.19)$$

In the case p = 1/2, the lines power vanishes leaving only continuum power = V^2.

Since the power is kept away from DC, Manchester coding is useful for AC-coupled transmission lines, such as lines incorporating transformers. The down side compared to NRZ is that the first spectral hump goes out to $\omega = 2\omega_1$, which reflects the fact that the minimum pulse width is $T_1/2$ whereas in NRZ it is T_1. So a transmission line must then have twice the bandwidth for Manchester relative to NRZ.

(f) Noise, ISI and Eye Patterns

In general, if some spectral components are filtered away in a transmission line (or in some general signal pathway), the corresponding pulse (by inverse Fourier Transform) has curved corners, meaning the pulse gets rounded and spread out. This effect along with noise can result in **inter-symbol interference (ISI)**. The superposition of such pulses on an oscilloscope (triggered on a recovered T_1 clock) for a random pulse train is called an **eye pattern.** This pattern must have a central clear area to allow the two (or more for some line codes) pulse levels to be distinguished by a receiving circuit. Here is a marginal eye pattern for NRZ on the left, and a better one for AMI on the right (see Section 37).

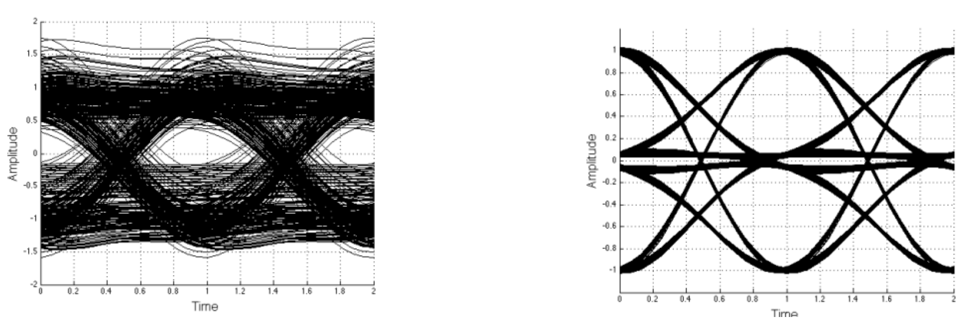


Fig 36.15

37. The AMI Line Code

(a) Pulse Shape

The pulse shape is the same as for unipolar NRZ ,

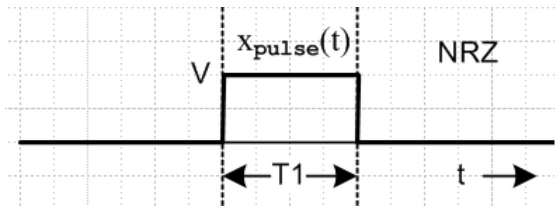


Fig 36.2

$$X_{pulse}(\omega) = (VT_1)\,\text{sinc}(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = (1/2\pi)\, V^2 T_1\, \text{sinc}^2(\omega T_1/2) \qquad (36.1)$$

However, we shall do the analysis below for a general $x_{pulse}(t)$ and insert the box shape at the end.

(b) Coding

Alternate Mark Inversion (AMI) means that a 0 is encoded as a zero (for duration T_1) and a 1 is encoded as a pulse (of duration T_1) of either plus or minus polarity. As each 1 is encountered in the data, the pulse polarity is the negative of that used for the previous encoded 1 pulse, so the 1 polarities are alternated, as in this example

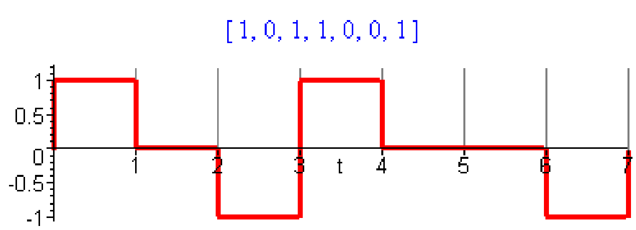


Fig 37.1

(c) Expectations $<y_m^2>$ and $<y_m y_n>$

A zero is coded with amplitude B = 0, but a one is coded with either A = +1 or A = -1, so we have A = ± 1, B = 0. Because there is now correlation between different locations m and n in the pulse train, we can no longer use the simple results of box (35.37). Consider then the expression given in (35.33) for the statistical average $<y_n^2>$. We assume that p is the probability of a 1 being coded, so 1-p is the probability of a 0 being coded. Then we have

$$<y_n^2> = [p]AA + [(1-p)]\,BB = [p]AA = [p]\,(\pm1)\,(\pm1) = p \qquad (37.1)$$

That was the easy one.

For $<y_n y_m>$ with m ≠ n, we have a *much* harder problem. Consider

$$<y_m y_n> = [pp]\, AA' + [p(1-p)]\, AB' + [(1-p)p]BA' + [(1-p)(1-p)]BB'$$

$$= [pp]\, \sigma\, \sigma' + [p(1-p)]\, \sigma\, 0 + [(1-p)p]\, 0\, \sigma' + [(1-p)(1-p)]\, 0\, 0$$

$$= p^2\, \sigma\, \sigma' \qquad (37.2)$$

where σ = ±1 and σ' = ±1. Here p^2 is the probability that both slot positions y_m and y_n are coded for 1. This can happen in four different ways, as illustrated here,

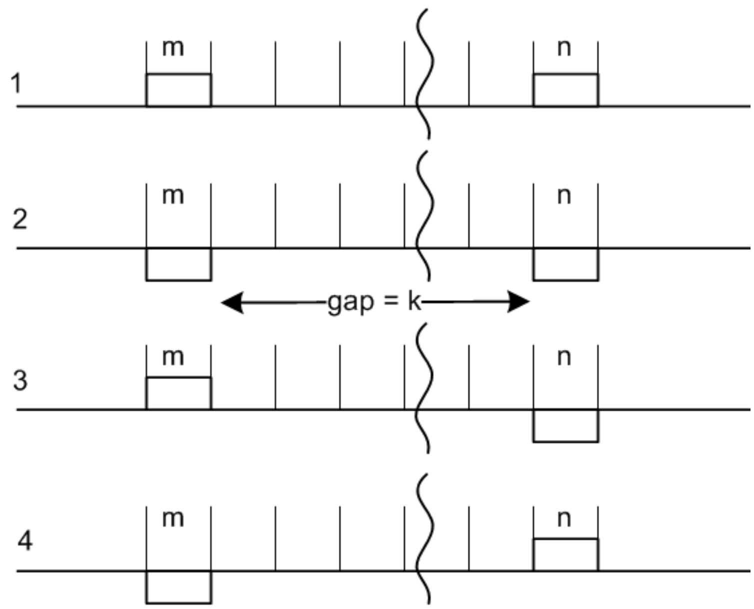


Fig 37.2

By symmetry, the probability of cases 1 and 2 is the same, and the probability of cases 3 and 4 is the same. This is perhaps not totally obvious, but the reason will become clear below when we talk about legal pulse patterns.

Given that both m and n are coded for 1, let X/2 be the total probability for the case 1, and Y/2 be the total for the case 3. Then we can write

$$<y_m y_n> = (+1)(+1)\, p^2X/2 + (-1)(-1)\, p^2X/2 + (+1)(-1)\, p^2Y/2 + (-1)(+1)\, p^2Y/2$$

$$= p^2(X-Y)\, .$$

Given that both m and n are coded for a 1, since we have enumerated all the cases, we must have

$X + Y = 1$ // probability of getting any of the four cases.

Our task then is to compute probabilities X and Y.

For cases 1 and 2 taken together, X is the probability that the gap between the coded 1's is filled with a legal sequence of pulses. This is the key statement and the reader may want to ponder the previous sentence thinking about probability as the number of legal ways divided by the total number of ways. Only the legal ways can show up in a statistical ensemble.

If the gap is "legal", there must be an *odd* number of coded 1's in the gap, due to the AMI alternation coding rule. Similarly, Y is the probability that there are an *even* number of coded 1's in the gap. Define,

$k = |m-n| - 1$ = size of gap

and think of X and Y as depending on k, so we write X_k and Y_k.

Note that $Y = Y_k = (1-X_k)$ = probability that gap has *even* number of coded 1's. So far, we add k labels to our results shown above,

$$\langle y_m y_n \rangle = p^2(X_k-Y_k) = p^2 (2X_k - 1) \qquad k = |m-n| - 1 \quad . \tag{37.3}$$

Assume we have a gap of size k and there exists some X_k and Y_k we don't yet know. What can be said about X and Y if the gap is increased to size k+1 by adding one more pulse period in between? Claim:

$X_{k+1} = Y_k\, p + X_k\, (1-p)$ = probability of having an odd number of coded 1's in gap k+1

Explanation:

• Y_k is the probability the k gap had an even number of coded 1's. In order to make the k+1 gap have an odd number of coded 1's we have to put a coded 1 in the new space, which has probability p.
This gives the first term $Y_k\, p$.

• X_k is the probability the k gap had an odd number of coded 1's. In order to make the k+1 gap have an odd number of coded 1's we have to put a coded 0 in the new space, which has probability (p-1).
This gives the second term $X_k\, (1-p)$.

Since this exhausts the ways we can get from k to k+1, X_{k+1} has the probability shown above. We could write a similar expression for Y_{k+1} but it is not needed. Since $Y_k = 1-X_k$ we then have

$$X_{k+1} = (1-X_k)\, p + X_k\, (1-p) = p - pX_k + X_k - pX_k = (1-2p)X_k + p \quad . \tag{37.4}$$

Now define,

$$a \equiv (1-2p) \qquad \Rightarrow \quad p = (1-a)/2 \qquad \text{and} \qquad 1-p = (1+a)/2$$

Then the above reads,

$$X_{k+1} = aX_k + p \ . \tag{37.5}$$

This is a difference equation (recurrence relation) which we want to solve for X_k. If there is no gap at all (k=0), we have two adjacent identical pulses which is illegal so $X_0 = 0$. If the gap is k = 1, then the middle element must be different from the two ends, so $X_1 = p$, consistent with (37.5). We now examine the recurrence relations:

$X_0 = 0$
$X_1 = p$
$X_2 = a(p) + p = p(a+1)$
$X_3 = a[p(a+1]+ p = p(a^2+a+1)$
....
$X_k = p\,(a^{k-1} + ... + a^2 + a + 1)$

The geometric series can be summed in the usual manner and yields

$$X_k = p\,(1 - a^k)/(1-a) = p\,(1 - a^k)/2p = (1 - a^k)/2\ . \qquad (37.6)$$

The same result can be obtained from Maple in this manner :

```
p := (1-a)/2:
Xk := rsolve({ X(k+1) = a*X(k) + p,X(0) = 0}, X): simplify(%);
```

$$-\frac{1}{2}a^k + \frac{1}{2}$$

Inserting this result into (37.3) gives

$$<y_m y_n> = p^2\,(2X_k - 1) = p^2\,(2[(1 - a^k)/2] - 1) = p^2\,(\,(1 - a^k) - 1) = - p^2 a^k$$

$$= - p^2\,a^{[|m-n|\ -\ 1]} = (-p^2/a)\,a^{|m-n|}\ . \qquad (37.7)$$

Therefore, we have our final results for our two expectations,

$$<y_m y_n> = (-p^2/a)\,a^{|m-n|} \qquad m \neq n \qquad // \ a \equiv (1-2p)$$
$$<y_n^2> = p\ . \qquad (37.8)$$

These results are in agreement with Bennett and Davey equations (19-111) and (19-119), and in fact it is their method we have presented above.

(d) The autocorrelation sequence

We assume the AMI code generator is stationary as defined in (35.10). Presumably this will be the case unless some non-stationary data stream is fed into the AMI encoder. With this assumption we can use (35.24) to relate the ensemble averages in (37.8) of our infinite sequence to the horizontal averages $<...>_1$,

$$r_s = \langle y_m y_{m+s}\rangle_1 = \langle y_m y_{m+s}\rangle \quad s\neq 0$$
$$r_0 = \langle y_m^2\rangle_1 = \langle y_m^2\rangle$$

where $r_s = \langle y_m y_{m+s}\rangle_1$ is the autocorrelation sequence for the y_m. Thus we have from (37.8)

$$r_s = (-p^2/a)\, a^{|s|} = -p^2 a^{|s|-1} = -p^2 (1-2p)^{|s|-1} \qquad s \neq 0$$
$$r_0 = p \tag{37.9}$$

We now plot the autocorrelation sequence for various p values using this code

```
restart; N := 6: with(plots):
col := array(0..10,[red,blue,green,brown,black,magenta,green,red,brown,black,gray]):
auto := proc(s,p) if (s=0) then p else -p^2*(1-2*p)^(abs(s)-1); fi; end:
a := 0: b := 5:
for n from a to b do
  p := n/10:  if p = 1/2 then p := 0.5000001; fi;  # avoid zero division
  for s from -N to N+1 do r[s] := auto(s,p); od; #autocorr data
  q := seq([s,r[s]],s=-N..N+1):  # create autocorr 2D point set
  g[n] := pointplot([q],color = col[n],style = line,thickness = 2, view =
-0.3..0.6,xtickmarks = 10);  # plot points and connect with lines
od:
display(seq(g[n],n=a..b)); # superpose the graphs for a through b
```

Here are plots of r_s for p = 0.0 (red) to p = 0.5 (black) where, as usual, we connect the discrete points of the sequence with lines,

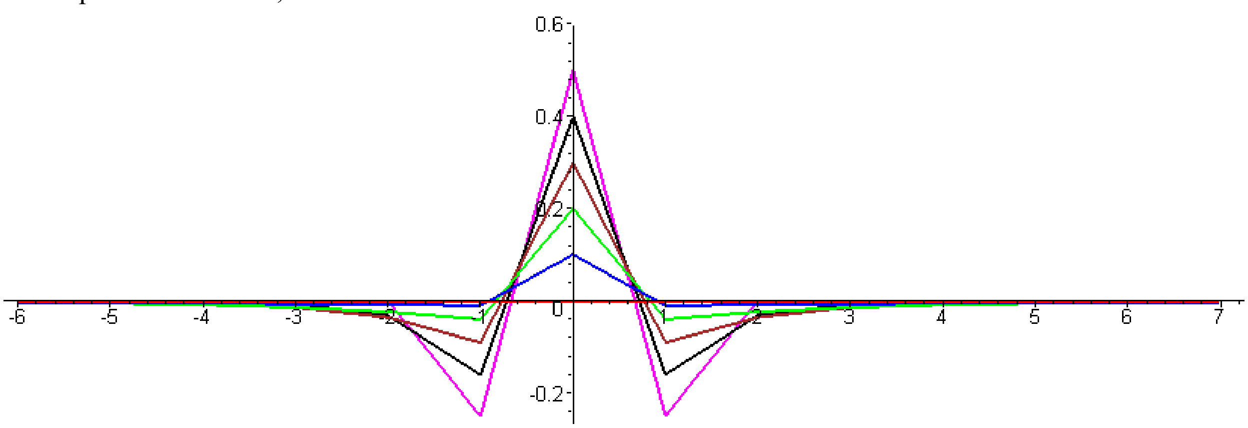


Fig 37.3

For p = 0, the autocorrelation sequence is a flat line (red) because in this case all symbols are 0. As the probability p of encoding a 1 increases, the autocorrelation sequence becomes more "active" away from the s = 0 central point. At p = 1/10 (blue), nothing much happens beyond |s| = 1. For p = 2/10 (green), we see action all the way out to |s| = 3 and even beyond. As p increases, the value of r_1 becomes larger and more negative, meaning if s = 0 is a coded + pulse, then s = 1 is likely to be a -1 pulse due to the AMI prescription. The p = 1/2 curve is magenta.

As p is *further* increased from p = 1/2, the density of coded 1's in the pulse train increases and the correlation distance increases as these 1's affect each other more and more. Here are plots for p = 0.5

(magenta) to p = 1.0 (gray). The action is so violent now that we had to increase the vertical range relative to the previous set of graphs.

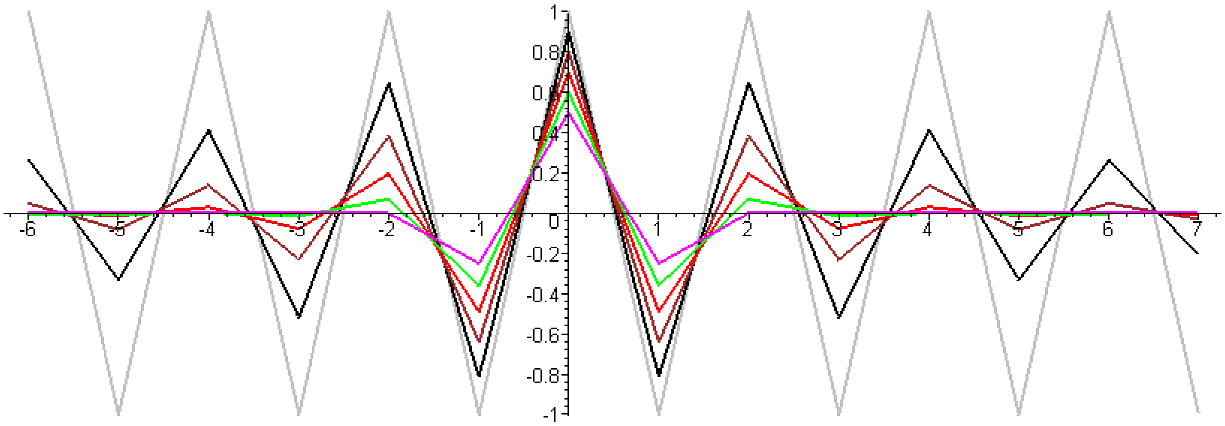


Fig 37.4

The p = 1 plot (gray) has a very simple interpretation: Now every pulse is coded as a 1, so every pulse must alternate in polarity, so as we slip two pulse trains relative to each other to obtain the autocorrelation sequence, if we slip an even number of symbols things are 100% correlated, and if we slip an odd number, things are then 100% anti-correlated.

(e) Power Spectral Density Calculation using the Autocorrelation Method

As outlined in Section 35, we have two methods to find the power spectrum. Here we shall use the "autocorrelation method" where R"(z) is the Z Transform of r_s .

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z) \qquad z = e^{i\omega T_1} \qquad (34.14a)$$

Our task then is to compute R"(z) from the autocorrelation sequence r_s given in (37.9)

$$r_s = (-p^2/a)\, a^{|s|} = -p^2\, a^{|s|-1} = -p^2\,(1-2p)^{|s|-1} \qquad s \neq 0$$
$$r_0 = p \qquad (37.9)$$

So:

$$R''(z) = \sum_{s=-\infty}^{\infty} r_s z^{-s} = p - (p^2/a) \sum_{s\neq 0} a^{|s|} z^{-s} \qquad |a|<1 \qquad a = 1-2p$$

$$= p - (p^2/a)[\sum_{s=1}^{\infty} a^s z^{-s} + \sum_{s=1}^{\infty} a^s z^s]$$

$$= p - (p^2/a)[\sum_{s=1}^{\infty} (a/z)^s + \sum_{s=1}^{\infty} (az)^s] \qquad |a/z| < 1 \qquad |az| < 1$$

$$= p - (p^2/a)[\sum_{s=1}^{\infty} \{x\}^s + \sum_{s=1}^{\infty} \{x^*\}^s] \qquad x = a/z \qquad |x| < 1 \quad |x^*|<1$$

$$= p - (p^2/a)[\frac{x}{1-x} + \frac{x^*}{1-x^*}] \qquad \text{// the usual geometric series sums}$$

$$= p - (p^2/a)\, 2\, \mathrm{Re}[\frac{x}{1-x}] \,. \qquad (37.10)$$

Now setting $\omega T_1 = k$ we have $z = e^{ik}$ and $x = a/z = a\, e^{-ik}$

$$\frac{x}{1-x} = \frac{ae^{-ik}}{(1- ae^{-ik})}\frac{(1- ae^{ik})}{(1- ae^{ik})} = \frac{a\,(e^{-ik}- a)}{1\text{-}2a\cos(k) + a^2} \qquad \Rightarrow \qquad \mathrm{Re}[\frac{x}{1-x}] = \frac{a\,(\cos(k)- a)}{1\text{-}2a\cos(k) + a^2}$$

and therefore

$$\mathrm{Re}[\frac{x}{1-x}] = \frac{a\,(\cos(\omega T_1)- a)}{1 + a^2 -2a\cos(\omega T_1)} \qquad a \equiv (1\text{-}2p) \qquad p = (1\text{-}a)/2 \qquad (37.11)$$

and then from (37.10),

$$R''(z) = p - (p^2/a)\, 2\,\frac{a\,(\cos(\omega T_1)- a)}{1 + a^2 -2a\cos(\omega T_1)} = p - 2p^2\frac{\cos(\omega T_1)- a}{1 + a^2 -2a\cos(\omega T_1)} \,. \qquad (37.12)$$

We then let Maple work on this expression a bit,

```
d := 1+a^2-2*a*cos(k):
n := cos(k)-a:
R := p - 2*p^2*(n/d);
```

$$R := p - 2\frac{p^2(\cos(k) - a)}{1 + a^2 - 2\,a\cos(k)}$$

```
p := (1-a)/2:
factor(simplify(R));
```

$$\frac{1}{2}\frac{(-1+a)(a+1)(-1+\cos(k))}{1+a^2-2\,a\cos(k)}$$

from which we learn that

$$R''(z) = \frac{1}{2}\,\frac{(1\text{-}a^2)(1\text{-}\cos(\omega T_1))}{1 + a^2 - 2a\cos(\omega T_1)} = \frac{(1\text{-}a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \,. \qquad (37.13)$$

Therefore the **AMI power spectral density** is given by

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathrm{pulse}}(\omega)\,\frac{(1\text{-}a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \qquad a = (1\text{-}2p) \qquad (1\text{-}a^2) = 4p(1\text{-}p) \qquad (37.14)$$

which is valid for $|a| < 1$ which means $0 < p < 1$.

(f) Summary, Plot and Limits of the AMI Spectral Power Density

Using $x = \omega/\omega_1 = \omega T_1/(2\pi) = fT_1$ = (37.15) can be written in these alternate forms (the last form uses (33.24)),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{(1-a^2)\sin^2(\omega T_1/2)}{1 + a^2 - 2a\cos(\omega T_1)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p) \qquad \textit{(37.15)}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{(1-a^2)[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p) \qquad (37.15a)$$

$$P(f) = 2\pi P_{pulse}(f) \frac{(1-a^2)[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \qquad (1-a^2) = 4p(1-p) \qquad (37.15b)$$

$$P(f) = 4p(1-p)\,|\mathcal{X}_{pulse}(f)|^2\,(1/T_1) \frac{\sin^2(\pi f T_1)}{1+(1-2p)^2-2(1-2p)\cos(2\pi f T_1)} \qquad (37.15c)$$

where we recall from the text after (1.4) that $X(\omega) = \mathcal{X}(f)$. This last result (37.15c) agrees with Bennet and Davey (19-123) but they have a leading factor 8 instead of 4. This is because they regard the frequency range for f as $(0,\infty)$ instead of $(-\infty,\infty)$ so the left part of the spectrum is folded over to the right side giving them an extra factor of 2.

Note that the AMI spectrum is completely continuous, there is no discrete part at all.

Setting p = 1/2 gives a = 0 so (37.14) simplifies somewhat to give,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \sin^2(\pi x) \qquad p = 1/2 \ . \qquad (37.16)$$

Selecting a box of height V and width T_1 we have from (36.1)

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (VT_1)^2 \,\mathrm{sinc}^2(\pi \frac{\omega}{\omega_1})/(2\pi T_1) = (V^2/\omega_1)\,\mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \qquad \textit{(36.1)}$$

so that

$$\mathcal{P}(\omega) = V^2(1/\omega_1)\,\mathrm{sinc}^2(\pi x)\,\sin^2(\pi x) \qquad p = 1/2 \qquad x = \omega/\omega_1 \qquad (37.17)$$

$$P(f) = V^2 T_1 \mathrm{sinc}^2(\pi f T_1)\,\sin^2(\pi f T_1) \qquad p = 1/2 \qquad x = fT_1 \qquad (37.17)'$$

This last result agrees with Xiong (2.34) where the code is called AMI-NRZ.

Plot: Ignoring now the overall factor (V^2/ω_1) [or V^2T_1] we get this AMI p = 1/2 power spectrum

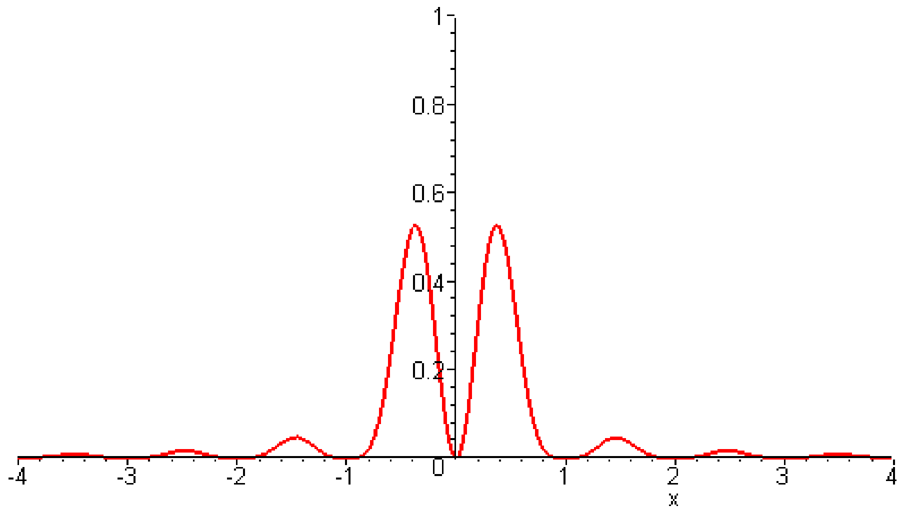


Fig 37.5

This shape is the same as the continuous part of the Manchester spectrum, but the first zero is at 1 instead of 2 since the pulse AMI pulse is twice as wide as the Manchester pulse.

The AMI Limit as $p \to 0$ ($a \to +1$)

Our derivation of (37.15) by either method required that $|a| < 1$ to obtain convergence of the geometric series, so we are a little wary of taking the limit $a \to 1$. We will find, however, that the limit gives the correct result so we must be getting convergence for the point a = 1 on the complex circle of convergence. The same comment applies to the limit $a \to -1$ of the next section.

In this limit p = 0, we know that our pulse train is just the constant value 0 so $\mathcal{P}(\omega) = 0$, so let's see how this happens from (37.15)

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega) \frac{(1-a^2)\,[\sin^2(\omega T_1/2)]}{1+a^2-2a\cos(\omega T_1)} = \mathcal{P}_{\mathtt{pulse}}(\omega)\,[\sin^2(\omega T_1/2)]\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)}$$

Using this limit from Appendix A

$$\lim_{a\to+1}\ (1/\pi)\,\frac{(1-a^2)}{1+a^2-2a\cos(2k)} = \sum_{m=-\infty}^{\infty} \delta(k - m\pi) \qquad \textit{(A.23c)}$$

we find that

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\,[\sin^2(\omega T_1/2)]\,\pi \sum_{m=-\infty}^{\infty} \delta(\omega T_1/2 - m\pi)$$

$$= \mathcal{P}_{\mathtt{pulse}}(\omega)\,\pi \sum_{m=-\infty}^{\infty} [\sin^2(m\pi)]\,\delta(\omega T_1/2 - m\pi) = 0 \qquad \text{since } \sin(m\pi) = 0 \text{ for all m}$$

The AMI Limit as p→ 1 (a → -1)

First of all, we can see that in this limit the AMI waveform has alternating-sign pulses. With V = 1, this waveform matches that shown in (34.21),

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega) \sum_{n=\pm odd} \delta(x - m/2) \qquad x = \omega/\omega_1 \quad . \qquad (34.21)$$

Somehow in this limit, the all-continuous AMI spectrum becomes all-discrete! How exactly does this happen? Consider again our continuous AMI result,

$$<\mathscr{P}(\omega)> = \mathscr{P}_{pulse}(\omega) \frac{(1-a^2)\,[\sin^2(\pi x)]}{1+a^2-2a\cos(2\pi x)} \qquad a = (1-2p) \quad . \qquad (37.15a)$$

It seems *possible* that this becomes discrete because when a = -1, $(1-a^2) = 0$ and $\mathscr{P}(\omega) = 0$ except possibly at singular points where the denominator vanishes. In Appendix A it is shown that

$$\lim_{a\to-1} \delta_8(k, a) = \lim_{a\to-1} (1/\pi) \frac{(1-a^2)\,[\sin^2(k)]}{1+a^2-2a\cos(2k)} = \sum_{m=\pm odd} \delta(k-m\pi/2) \quad . \qquad (A.25a)$$

Therefore we may write

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\, \pi\, \delta_8(\pi x,a)$$

$$\to \mathscr{P}_{pulse}(\omega)\, \pi \sum_{m=\pm odd} \delta(\pi x-m\pi/2) = \mathscr{P}_{pulse}(\omega) \sum_{m=\pm odd} \delta(x-m/2)$$

and this agrees with our expected result shown just above.

38. The Change/Hold Line Code

Here we mimic the previous Section on the AMI code. The calculation of $<y_m y_n>$ is similar, but not the same. At the end, we apply the results to the NRZI line code.

(a) Pulse Shape

The pulse shape is the same as for unipolar NRZ ,

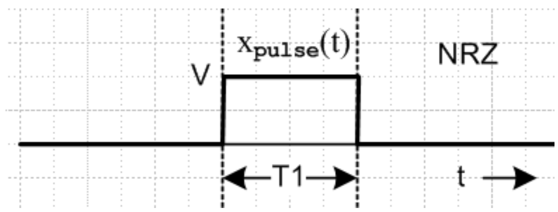


Fig 36.2

$$X_{pulse}(\omega) = (VT_1)\ \mathrm{sinc}(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = (1/2\pi)\ V^2 T_1\ \mathrm{sinc}^2(\omega T_1/2) \qquad (36.1)$$

In place of amplitude V we will have A and B as described below.

(b) Coding

The coding uses two amplitudes A and B. Hold or Change coding means that a 0 is encoded as no change in the pulse amplitude (it holds, remaining what it was), while a 1 is encoded as a change A↔B. Here is an example starting with an A pulse:

```
data   =     [ 1  0  1  1  0  0  1]
encode =  [ A  B  B  A  B  B  B  A]
```

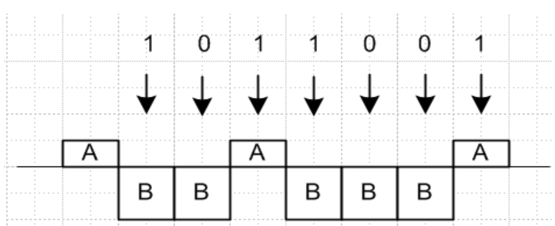


Fig 38.1

Again we shall assume an arbitrary pulse shape and insert the box-pulse at the end.

(c) Expectations $<y_m^2>$ and $<y_m y_n>$

First consider

$<y_n^2> = [q]A^2 + [(1-q)] B^2$,

where q is the probability that slot n has y_n = A. In this code, since only change and hold are coded, there is no preference for either amplitude, so q = 1/2 and

$$<y_n^2> = (A^2+B^2)/2 \quad . \qquad (38.1)$$

We define p to be the probability of a change (data = 1), and 1-p to be the probability of a hold (data = 0).

Turning to $<y_n y_m>$, consider this picture similar to that used for the AMI case, where the gap is kT_1units. Here we arbitrarily draw A > 0 and B < 0 and we draw the pulse as square, but it could be any shape and A and B can have any signs.

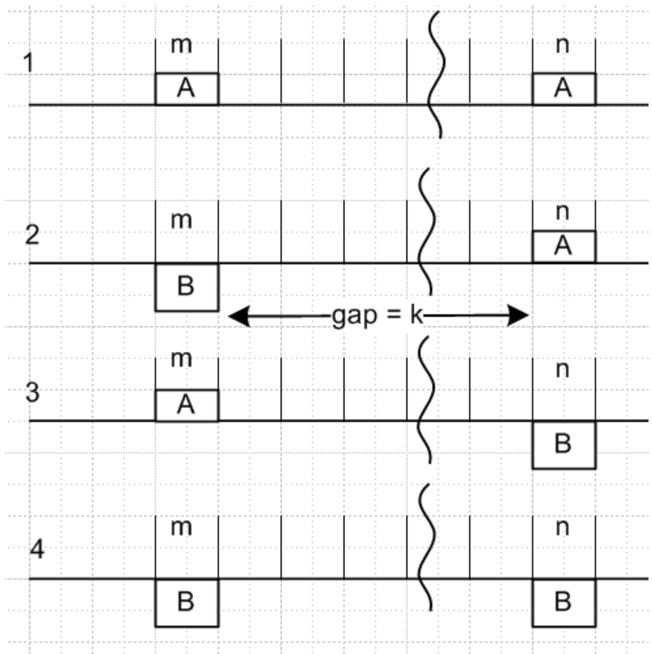


Fig 38.2

Denote the four probabilities as $p^{(AA)}{}_k$ and so on. Since slot m and slot n must each be filled with either an A or a B, this picture shows the only four possibilities, so (different scaling relative to AMI analysis)

$$p^{(AA)}{}_k + p^{(AB)}{}_k + p^{(BA)}{}_k + p^{(BB)}{}_k = 1 \quad .$$

Note that $p^{(AA)}{}_k$ is the probability of slot m and slot n both having amplitude A in the statistical pulse train. With these probabilities, we will have

$$<y_m y_n> = p^{(AA)}{}_k AA + p^{(AB)}{}_k AB + p^{(BA)}{}_k BA + p^{(BB)}{}_k BB \quad . \qquad (38.2)$$

In case 1, there are a certain number of holds and changes during the gap such that the overall effect is a hold. The number of changes must have been even. But this same statement can be made about case 4, so cases 1 and 4 have the same probability of existing in the pulse train. Similarly, cases 2 and 3 have the same probability and in those cases the number of changes must be odd. So now we have two variables to worry about and they add to 1/2 :

$$p^{(AA)}{}_k + p^{(AB)}{}_k = 1/2 \tag{38.3}$$

$$\langle y_m y_n \rangle = p^{(AA)}{}_k\, AA + p^{(AB)}{}_k\, AB + p^{(AB)}{}_k\, BA + p^{(AA)}{}_k\, BB$$

$$= p^{(AA)}{}_k\,(AA + BB) + p^{(AB)}{}_k\,(AB + BA)$$

so

$$\langle y_m y_n \rangle = p^{(AA)}{}_k(A^2 + B^2) + p^{(AB)}{}_k\, 2AB \quad . \tag{38.4}$$

If the gap is zero, what is the probability of having an adjacent AA in the pulse stream? The probability of having the left A is 1/2, and the probability for an A being followed by a A is 1-p. Therefore

$$p^{(AA)}{}_0 = (1/2)(1\text{-}p) \quad . \tag{38.5}$$

Consider now the gap as shown at value k. We claim that

$$p^{(AA)}{}_{k+1} = p^{(AA)}{}_k\,(1\text{-}p) + p^{(AB)}{}_k\, p \quad . \tag{38.6}$$

<u>Proof</u>: If it was an AA to start with gap k, then to be AA with gap k+1 we have to add another A which has probability (1-p) since this is a hold. Conversely, if it was an AB we have to add an A which is a change, which has probability p. Then from (38.3) we rewrite (38.6) as

$$p^{(AA)}{}_{k+1} = p^{(AA)}{}_k\,(1\text{-}p) + (1/2 - p^{(AA)}{}_k\,)\, p \quad . \tag{38.7}$$

To simplify notation, let $X_k \equiv p^{(AA)}{}_k$ so that $p^{(AB)}{}_k = 1/2 - X_k$. Then (38.4) and (38.7) become

$$\langle y_m y_n \rangle = X_k\,(A^2 + B^2) + (1/2 - X_k)2AB = (A\text{-}B)^2 X_k + AB \tag{38.8}$$

$$X_{k+1} = X_k\,(1\text{-}p) + (1/2\text{-}X_k)\,p = (1\text{-}2p)X_k + p/2$$
$$= aX_k + p/2 \qquad\qquad a \equiv 1\text{-}2p \text{ , same as for AMI} \tag{38.9}$$

Maple solves this recursion equation as follows, using (38.5) that $X_0 = (1/2)(1\text{-}p)$,

```
p := (1-a)/2:
Xk := rsolve({ X(k+1) = (1-2*p)*X(k) + p/2, X(0) = (1-p)/2}, X):
simplify(%);
```

$$\frac{1}{4}a^{(k+1)} + \frac{1}{4}$$

so we find that

$$X_k = (1 + a^{k+1})/4 = p^{(AA)}{}_k \quad . \tag{38.10}$$

As a check, suppose p = 0 so there can be no changes. Then a = 1 and $p^{(AA)}{}_k = 1/2$. We can now have only case 1 or case 4, so we know $p^{(AB)}{}_k = 0$, and that is consistent with $p^{(AA)}{}_k + p^{(AB)}{}_k = 1/2$.

Continuing from (38.8),

$$\langle y_m y_n \rangle = (A-B)^2 X_k + AB = (A-B)^2(1 + a^{k+1})/4 + AB$$

$$= [\, (A-B)^2/4]\, a^{k+1} + (A-B)^2/4 + AB$$

$$= [\, (A-B)^2/4]\, a^{k+1} + [(A+B)^2/4]$$

$$= (1/4)\, [\, (A-B)^2\, a^{k+1} + (A+B)^2\,] \qquad (38.11)$$

and we note that the result is indeed symmetric under A↔ B. Again for p = 0 (always hold, a=1) we find that $\langle y_m y_n \rangle = (A^2+B^2)/2$ which is the same then as $\langle y_n^2 \rangle$. For a constant pulse train, the amount of slot separation makes no difference.

Since k = |m-n| - 1 in general, we get these final results,

$$\langle y_m y_n \rangle = (1/4)\, [\, a^{|m-n|}\, (A-B)^2 + (A+B)^2\,] \qquad m \neq n \qquad a \equiv 1-2p$$

$$\langle y_n^2 \rangle = (A^2+B^2)/2 \ . \qquad (38.12)$$

(d) The autocorrelation sequence

We assume the Change/Hold code generator is stationary as defined in (35.10). Presumably this will be the case unless some non-stationary data stream is fed into the Change/Hold encoder. With this assumption we can use (35.24) to relate the ensemble averages in (37.8) of our infinite sequence to the horizontal averages $\langle ... \rangle_1$,

$$r_s = \langle y_m y_{m+s} \rangle_1 = \langle y_m y_{m+s} \rangle \quad s \neq 0$$
$$r_0 = \langle y_m^2 \rangle_1 = \langle y_m^2 \rangle$$

where $r_s = \langle y_m y_{m+s} \rangle_1$ is the autocorrelation sequence for the y_m. Thus we have from (38.12)

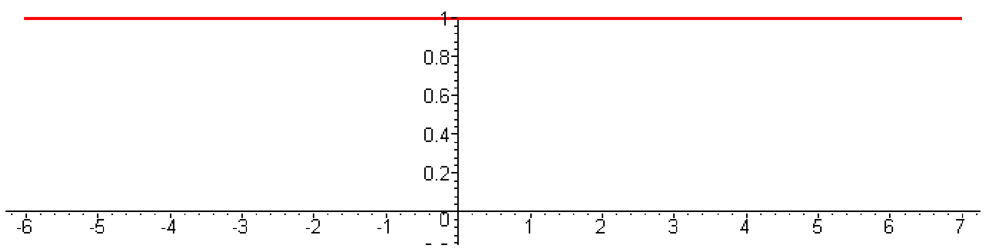
$$r_s = (1/4)\, [\, a^{|s|}\, (A-B)^2 + (A+B)^2\,] \qquad s \neq 0 \qquad a \equiv 1-2p$$
$$r_0 = (A^2+B^2)/2 \ . \qquad (38.13)$$

Note that for p = 0, we get $r_s = (1/4)\, [\, 1^{|s|}\, (A-B)^2 + (A+B)^2\,] = (A^2+B^2)/2 = r_0$, so in this case the autocorrelation plot will be a horizontal line (red below) at this value $(A^2+B^2)/2$. This is the case of never any change, so half our ensemble is all A and half is all B which is why $\langle y_m^2 \rangle = (A^2+B^2)/2$.

We now plot the autocorrelation function for various p values using this code

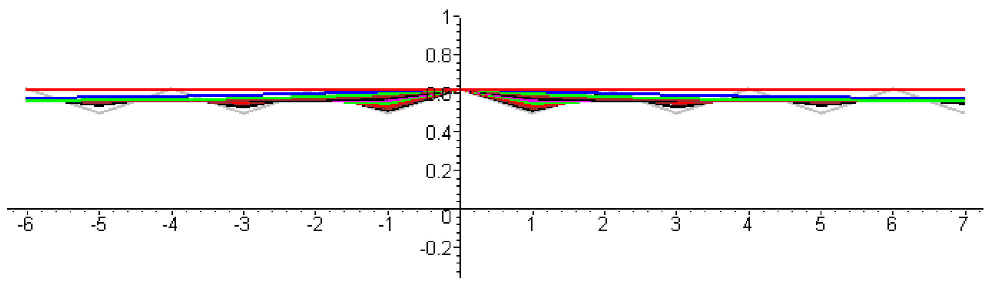
```
restart; N := 6: with(plots): A := 1:  B := 1/2:
col := array(0..10,[red,blue,green,brown,black,magenta,green,red,brown,black,gray]):
auto := proc(s,p) if (s=0) then (A^2+B^2)/2;
  else (1/4)*( (1-2*p)^abs(s)*(A-B)^2 + (A+B)^2 ); fi; end:
a := 0: b := 10:
for n from a to b do
  p := n/10:  if p = 1/2 then p := 0.5000001; fi;  # avoid zero division
  for s from -N to N+1 do r[s] := auto(s,p); od; #autocorr data
  q := seq([s,r[s]],s=-N..N+1):  # create autocorr 2D point set
  g[n] := pointplot([q],color = col[n],style = line,thickness = 2, view =
-1..1,xtickmarks = 10);  # plot points and connect with lines
od:
display(seq(g[n],n=a..b)); # superpose the graphs for a through b
```

For all plots below, we assume A = 1, and we plot curves for p = 1/n for n = 0 to 10 giving

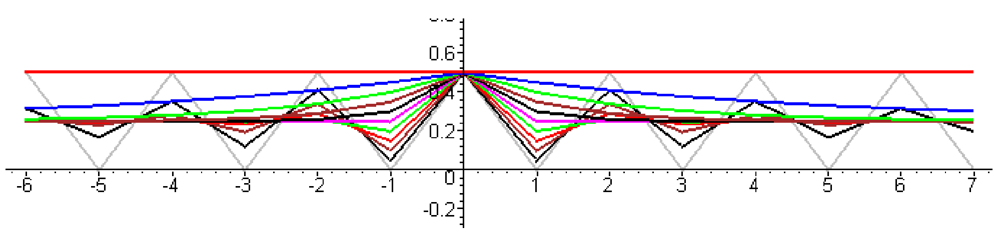


B = 1 (a)

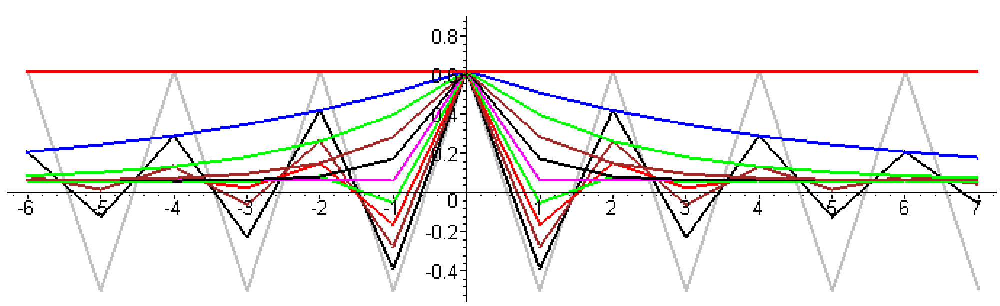
Since B = A = 1, there is no change whether or not we "hold" or change", so all plots are the same.



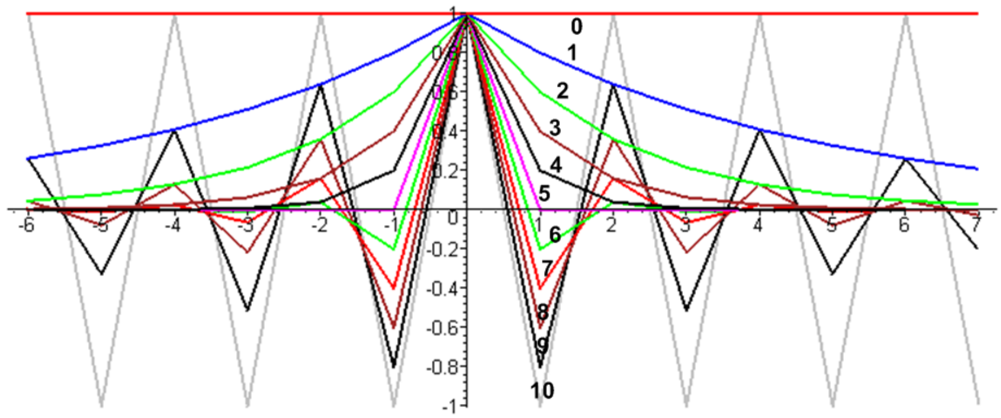
B = 1/2 (b)



B = 0 (c)



B = -1/2 (d)



B = -1 (e)

[label **n** means p = n/10] Figures 38.3 (a) through (e)

For each set of plots other than the first, as p ranges from 0 to 1/2, the infinite range correlation of having no change (red curve, p = 0) reduces (underdamped, shall we say), and for p = 1/2 we have a constant value for $|s| \geq 1$ (critically damped). Then for p > 1/2 we have "oscillation" (overdamped). The ultimate case occurs when p = 1 where there the values A = 1 and B = -1 strictly alternate, so we have a square wave. As we relatively slide a pair of these square waves to create the autocorrelation function, as expected correlation jumps between +1 for even symbol shifts and -1 for odd symbol shifts (gray curve).

(e) Power Spectral Density Calculation using the Autocorrelation Method

As outlined in Section 35, we have two methods to find the power spectrum. Here we shall use the "autocorrelation method" where R"(z) is the Z Transform of r_s .

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z) \qquad z = e^{i\omega T_1} \tag{34.14a}$$

Our task then is to compute R"(z) from the autocorrelation sequence r_s given in (38.13)

$$r_s = (1/4)\,[\, a^{|s|}\,(A-B)^2 + (A+B)^2\,] \qquad s \neq 0 \qquad a \equiv 1-2p$$
$$r_0 = (A^2+B^2)/2 \ . \tag{38.13}$$

So:

$$R''(z) = \sum_{s=-\infty}^{\infty} r_s\, z^{-s} = (A^2+B^2)/2 + (1/4)\,(A-B)^2 \sum_{s \neq 0} a^{|s|}\, z^{-s} + (1/4)\,(A+B)^2 \sum_{s \neq 0} z^{-s} \tag{38.14}$$

The two sums have (conveniently) already been computed:

$$\sum_{s \neq 0} a^{|s|}\, z^{-s} = 2\,\frac{a\,(\cos(\omega T_1) - a)}{1 + a^2 - 2a\cos(\omega T_1)} \qquad |a| < 1 \qquad (=2\,\mathrm{Re}[\frac{x}{1-x}]) \qquad (37.10) \text{ and } (37.11)$$

$$\sum_{s \neq 0} z^{-s} = \sum_{s=-\infty}^{\infty} z^{-s} - 1 = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) - 1 \qquad z = e^{i\omega T_1} \qquad \textit{(13.2) with } k = \omega T_1$$

Thus

$$R''(z) = (A^2+B^2)/2 + (1/4)\,(A-B)^2\,[2\,\frac{a\,(\cos(\omega T_1)- a)}{1 + a^2 -2a\cos(\omega T_1)}\,] + (1/4)\,(A+B)^2[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) - 1]$$

$$= (1/4)\,(A-B)^2 + (1/4)\,(A-B)^2\,[2\,\frac{a\,(\cos(\omega T_1)- a)}{1 + a^2 -2a\cos(\omega T_1)}\,] + (1/4)\,(A+B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (1/4)\,(A-B)^2\,[\,1 + 2\,\frac{a\,(\cos(\omega T_1)- a)}{1 + a^2 -2a\cos(\omega T_1)}\,] + (1/4)\,(A+B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (1/4)\,(A-B)^2\,\frac{(1-a^2)}{1 + a^2 - 2a\cos(\omega T_1)} + (1/4)\,(A+B)^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \qquad (38.15)$$

Installing this result into (34.14a) which says $\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,R''(z)$, we get a final result for the power **spectral density of the Change/Hold line code**:

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\{\,[\frac{(A-B)}{2}\,]^2\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}\,]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\,\} \qquad (38.16)$$

$$a = (1-2p) \qquad (1-a^2) = 4p(1-p)$$

(f) Summary, Limits and Plot of the Change/Hold Spectral Power Density

We shall now investigate various limits of our final result which was (38.16),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\{\,[\frac{(A-B)}{2}\,]^2\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}\,]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\,\} \qquad (38.17)$$

<u>Limit A → B</u>: Setting A = B in (38.17) gives

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\{\,A^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1-2\pi m)\,\} \quad . \qquad (38.18)$$

In the case that the pulse is a box of unit height we use (34.22),

$$\mathcal{P}_{pulse}(\omega) = (1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2) \qquad (38.19)$$

to get

$$\mathcal{P}(\omega) = A^2 \mathcal{P}_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)$$

$$= A^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\, (1/\omega_1)\, \mathrm{sinc}^2(\pi m) = A^2\, 2\pi\, \delta(\omega T_1)\, (1/\omega_1)$$

$$= A^2\, \delta(\omega)\ . \tag{38.20}$$

This is exactly what we expect when A = B, since the pulse train is then just a constant value A !

<u>Limit $p \rightarrow 1$</u> ($a \rightarrow -1$) :

In this limit we expect to get a square-wave pulse train with alternating values A and B. We make use of this limit from Appendix A with $2k = \omega T_1$,

$$\lim_{a\rightarrow -1} \frac{(1-a^2)}{1+a^2 - 2a\cos(\omega T_1)} = \pi \sum_{m=\pm odd} \delta(\omega T_1/2 - m\pi/2) = \sum_{m=\pm odd} 2\pi\, \delta(\omega T_1 - m\pi) \qquad \textit{(A.25b)}$$

and then the Change/Hold spectral power density (38.17) becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left\{ \left[\frac{(A-B)}{2}\right]^2 \sum_{m=\pm odd} 2\pi\, \delta(\omega T_1 - m\pi) + \left[\frac{(A+B)}{2}\right]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right\} .$$

Installing from (34.22) the unit-height box pulse shape $\mathcal{P}_{pulse}(\omega) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2)$ and using (38.20) the second term becomes just $\left[\frac{(A+B)}{2}\right]^2 \delta(\omega)$ while the first term is

$$\left[\frac{(A-B)}{2}\right]^2 \sum_{m=\pm odd} 2\pi\, \delta(\omega T_1 - m\pi)\, (1/\omega_1)\, \mathrm{sinc}^2(m\pi/2)$$

$$= \left[\frac{(A-B)}{2}\right]^2 2\pi \sum_{m=\pm odd} \delta(\omega T_1 - m\pi)\, (1/\omega_1)\, (m\pi/2)^{-2}$$

$$= (A-B)^2\, (1/\pi^2) \sum_{m=\pm odd} (1/m^2)\, \delta(\omega - m\omega_1/2)$$

giving a final result,

$$\mathcal{P}(\omega) = \left[\frac{(A-B)}{\pi}\right]^2 \sum_{m=\pm odd} (1/m^2)\, \delta(\omega - m\omega_1/2) + \left[\frac{(A+B)}{2}\right]^2 \delta(\omega)\ . \tag{38.21}$$

Comparing the first term with (34.23), we see that it is the spectrum of a square wave whose peak-to-peak amplitude is (A-B), which is exactly what it should be since every pulse is a "change". The second term then correctly accounts for the expected average DC level of (A+B)/2.

Limit p→0 (a → +1) :

In this case for a square wave we expect to get a result appropriate for an ensemble of pulse trains half of which have constant value A and the other have constant value B, since all pulse trains are in a permanent hold state with p = 0; nothing changes. This time we use this limit (A.23c) with $2k = \omega T_1$,

$$\lim_{a\to+1} \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} = \pi \sum_{m=-\infty}^{\infty} \delta(\omega T_1/2 - m\pi) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - m2\pi) \qquad \textit{(A.23c)}$$

to get from (38.21),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \{ [\frac{(A-B)}{2}]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2m\pi) + [\frac{(A+B)}{2}]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \}$$
$$= \frac{A^2+B^2}{2} \mathcal{P}_{pulse}(\omega) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2m\pi) \ . \qquad (38.22)$$

Ignoring the leading factor, this agrees with the first line of (33.25) which was developed for a simple pulse train with unit amplitudes $y_n = 1$. This result then describes the average spectral power of an ensemble in which 50% of the pulse trains have amplitude A and the rest amplitude B.

Installing the square pulse spectrum (34.22) $\mathcal{P}_{pulse}(\omega) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2)$ gives

$$\mathcal{P}(\omega) = \frac{A^2+B^2}{2} (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2m\pi)$$
$$= \frac{A^2+B^2}{2} (1/\omega_1)\, 2\pi\, \delta(\omega T_1) = \frac{A^2+B^2}{2} (T_1\omega_1)^{-1}\, 2\pi\, \delta(\omega)$$
$$= \frac{A^2+B^2}{2} \delta(\omega)$$

which describes an ensemble of constant pulse trains 50% of which are x(t) = A and the rest x(t) = B.

Limit p→1/2 (a → 0)

Recall again the general result (38.17),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \{ [\frac{(A-B)}{2}]^2 \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \qquad \textit{(38.17)}$$

The big ratio becomes unity so the spectral power density is then

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \{ [\frac{(A-B)}{2}]^2 + [\frac{(A+B)}{2}]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \ . \qquad (38.23)$$

Box-Shaped Pulse for general p: Start as just above with (38.171) and insert $\mathcal{P}_{\mathbf{pulse}}(\omega)$ for the box,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left\{ \left[\frac{(A-B)}{2}\right]^2 \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \left[\frac{(A+B)}{2}\right]^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right\}$$

$$\mathcal{P}_{\mathbf{pulse}}(\omega) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) \ .$$

As usual, the second term becomes $\left[\frac{(A+B)}{2}\right]^2 \delta(\omega)$, so the result is [$a = 1-2p$, $x = \omega/\omega_1 = fT_1$]

$$\mathcal{P}(\omega) = \left[\frac{(A-B)}{2}\right]^2 (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) \frac{(1-a^2)}{1+a^2 -2a\cos(\omega T_1)} + \left[\frac{(A+B)}{2}\right]^2 \delta(\omega) \qquad (38.24)$$

$$P(f) = \left[\frac{(A-B)}{2}\right]^2 T_1\, \mathrm{sinc}^2(\pi f T_1) \frac{(1-a^2)}{1+a^2 -2a\cos(2\pi f T_1)} + \left[\frac{(A+B)}{2}\right]^2 \delta(f) \ . \qquad (38.24)'$$

Box-Shaped Pulse for p = 1/2 (a = 0):

$$\mathcal{P}(\omega) = \left[\frac{(A-B)}{2}\right]^2 (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) + \left[\frac{(A+B)}{2}\right]^2 \delta(\omega)$$

$$= (1/\omega_1) \left\{ \left[\frac{(A-B)}{2}\right]^2 \mathrm{sinc}^2(\pi x) + \left[\frac{(A+B)}{2}\right]^2 \delta(x) \right\} \quad x \equiv \frac{\omega}{\omega_1} = f\,T_1 \qquad (38.25)$$

$$P(f) = T_1 \left[\frac{(A-B)}{2}\right]^2 \mathrm{sinc}^2(\pi f T_1) + \left[\frac{(A+B)}{2}\right]^2 \delta(f) \ . \qquad (38.25)'$$

Ignoring the factor $(1/\omega_1)$ in (38.25), we make this plot of $\mathcal{P}(\omega)$, which is the same as for unipolar NRZ but with different scaling factors for the two terms,

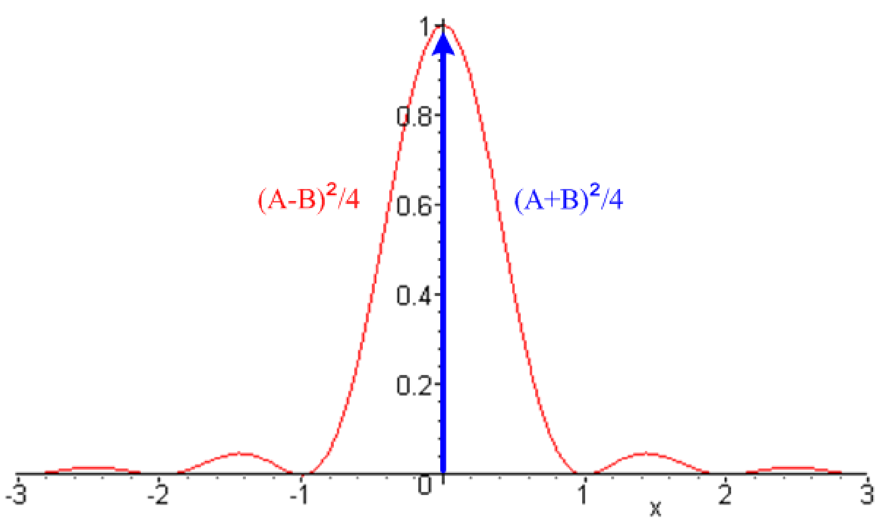


Fig 38.4

Example 1: Unipolar NRZI line code

Coding: This is a special case of Change/Hold encoding where A = 1 and B = 0.

```
data   =     [ 1  0  1  1  0  0  1]
encode =  [ 1  0  0  1  0  0  0  1]
```

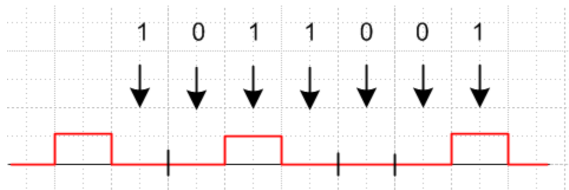


Fig 38.5

NRZI means NRZ Invert-on-1, where NRZ means non-return to zero (see comments at the start of Section 36). NRZI does not mean "NRZ inverted". Some other sources use A = 0 and B = 1 so then transitions happen on 0 instead of 1, as in the standard for USB (Universal Serial Bus). The Change/Hold spectra are symmetric in A↔B, so the NRZI spectra are the same for either convention.

Expectation Values

$$<y_m y_n> = (1/4)\,[\, a^{|m-n|}\,(A-B)^2 + (A+B)^2\,] = (1/4)\,[\, a^{|m-n|} + 1\,]$$

$$<y_n^2> = (A^2+B^2)/2 = 1/2 \qquad (38.26)$$

Spectrum: From (38.17), and with a = (1-2p),

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\{\, [\frac{(A-B)}{2}]^2 \frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + [\frac{(A+B)}{2}]^2 \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\}$$

$$= \mathcal{P}_{pulse}(\omega)\,\{\, \frac{1}{4}\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \frac{1}{4}\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\,\}\,. \qquad (38.27)$$

Box-Shaped Pulse for general p: From (38.24),

$$\mathcal{P}(\omega) = \frac{1}{4}\,(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2)\,\frac{(1-a^2)}{1+a^2-2a\cos(\omega T_1)} + \frac{1}{4}\,\delta(\omega)\,. \qquad (38.28)$$

Box-Shaped Pulse for p = 1/2 (a = 0): From (38.25),

$$\mathcal{P}(\omega) = (1/\omega_1)\,\{\, \frac{1}{4}\,\mathrm{sinc}^2(\pi x) + \frac{1}{4}\,\delta(x)\,\} \qquad x \equiv \frac{\omega}{\omega_1} = fT_1 \qquad (38.29)$$

$$P(f) = T_1\,\frac{1}{4}\,\mathrm{sinc}^2(\pi x) + \frac{1}{4}\,\delta(f)\,. \qquad (38.29)'$$

The plot is that of Fig 38.3, but with each factor being 1/4.

The spectrum (38.29) is exactly the same as that for unipolar NRZ shown in (36.3) with V = 1. One way to understand this fact is that for every NRZ sequence y_n there is an NRZI sequence y'_n ,

$$y'_n = y_n - y_{n-1} \quad . \qquad // \text{ mod-2 math}$$

This equation can be solved for y_n in terms of y'_n (assume y_0= 0),

$$y_n = \Sigma_{m=1}^{n} y'_m \quad n = 1,2,3....$$

Consider the space of all random sequences of 1's and 0's (random pulse trains p = 1/2). Since we just showed that the relation $\{y_n\} \leftrightarrow \{y'_n\}$ is one-to-one, the mapping f: $\{y_n\}\rightarrow\{y'_n\}$ just reorders the set of random sequences in the ensemble used to compute the spectral power density, so that density cannot change.

In contrast, for $p \neq 1/2$, the NRZI power spectrum (38.28) is quite different from the NRZ power spectrum (36.3),

$$\mathscr{P}(\omega) = \frac{1}{4} (1/\omega_1) \left[\text{sinc}^2(\pi x) \frac{(1-a^2)}{1+a^2 -2a\cos(\omega T_1)} + \delta(x)\right] \qquad // \text{ unipolar NRZI} \qquad (38.28)$$

$$a = (2p-1)$$

$$\mathscr{P}(\omega) = \frac{1}{4} (1/\omega_1) \left[\text{sinc}^2(\pi x)(1-a^2) + (2p)^2 \delta(x)\right] \qquad // \text{ unipolar NRZ} \qquad (36.3)$$

Example 2: Bipolar NRZI line code

```
data   =     [ 1  0  1  1  0  0  1]
encode =  [ 1 -1 -1  1 -1 -1 -1  1]
```

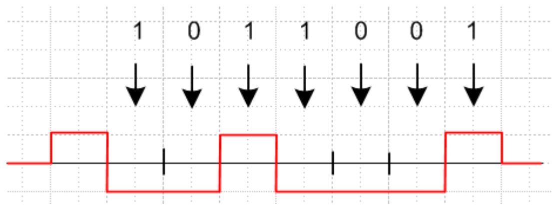


Fig 38.6

In this case A = 1 and B = -1 so the general Change/Hold spectrum (38.17) becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \frac{(1-a^2)}{1+a^2 -2a\cos(\omega T_1)} \qquad a = (2p-1) \qquad \text{// bipolar NRZI} \qquad (38.30)$$

and for p = 1/2 (a=0) we obtain,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \ . \qquad \text{// } = (1/2\pi)\ T_1 \ \mathrm{sinc}^2(\omega T_1/2) \ \text{ for the box pulse (36.1)} \qquad (38.31)$$

In contrast, the Bipolar NRZ spectrum is the general result shown in box (35.37)

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\right] \qquad \textit{(35.28)}$$

with (as shown below Fig 36.5) $\sigma^2 = 4p(1-p) = (1-a^2)$ and $\mu^2 = (1-2p)^2 = a^2$, so

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[(1-a^2) + a^2 \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)\right] \qquad \text{// bipolar NRZ} \qquad (38.32)$$

If p = 1/2 (a = 0) then the Bipolar NRZ spectrum becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathbf{pulse}}(\omega) \qquad (38.33)$$

which is the same as for Bipolar NRZI.

Appendix A: Delta Function Technology

The delta function is a mathematical tool that simplifies the description of mathematical relationships and allows one to consider certain idealized situations which cannot really exist in practice. The down side is that delta functions also serve to confuse readers who are not used to working with them. As later sections of this monograph were written, it became apparent that much of the development leans fairly heavily on "delta function technology", and one must understand delta functions at a somewhat deeper level than presented in Section 1. Of particular importance for spectral power density is our special meaning for the symbol $\delta(0)$ which appears in Chapter 6.

Early authors were squeamish about using the Dirac delta function (for example, Smythe). The theory of distributions was made rigorous in the 1935-1940 era by Sobolev and then Schwartz. The theory makes use of a class of extremely smooth functions and operators called linear functionals, and the reader can find a good presentation in Stakgold Chapters 1 and 5.

In brief, a functional T is just a mapping from some Hilbert Space to the real numbers. Usually that Hilbert Space is a set of reasonable functions defined on some interval. One can write a functional T in this notation: $\langle T,f\rangle$ = real number, some function of T and f. A functional is linear if it does the usual things like $\langle T,f_1+f_2\rangle = \langle T,f_1\rangle + \langle T,f_2\rangle$.

A distribution t is any linear functional which acts on a certain subset of all possible functions f. It acts (theoretically) only on the subset of smooth functions φ called test functions. Thus, a distribution t is represented as $\langle t,\varphi\rangle$ = real number, a function of t and φ.

For example, every reasonable real function f defines a linear functional and thus a distribution in this manner, where we happen to use the interval $(-\infty,\infty)$,

$$\langle f,\varphi\rangle = \int_{-\infty}^{\infty} dx\, f(x)\varphi(x) \ .$$

Obviously the integral of real functions is some real number. The function f(x) must be reasonable enough so that the integral is well behaved (it is "locally integrable").

In this context, the linear functional which defines the distribution known as the delta function is given as

$$\langle \delta_\xi,\varphi\rangle = \varphi(\xi) \qquad \text{special case:} \quad \langle \delta,\varphi\rangle = \varphi(0)$$

or

$$\int_{-\infty}^{\infty} dx\, \delta(x-\xi)\varphi(x) = \varphi(\xi) \qquad\qquad \int_{-\infty}^{\infty} dx\, \delta(x)\varphi(x) = \varphi(0)$$

which we normally think of as the sifting property (2.3). The thing δ_ξ is the distribution (the linear functional name), while the thing written as $\delta(x-\xi)$ is a "symbolic function" or "generalized function" associated with the distribution δ_ξ. But loosely speaking, one refers to $\delta(x-\xi)$ as the distribution.

The general idea is that functions like $\delta(x)$ or $\delta'(x)$ have a meaning only when they are inside an integral. When they appear standing alone, they are "symbolic functions", such as in (4.6)
$[\ RC\ d/dt + 1]\ g(t) = \delta(t)$. This is a symbolic or distributional equation which acquires meaning when both sides are placed inside the same integration. Formally that integration should be against a test function, but we can think of it as being against any reasonable function; the main idea is that everything must converge.

Although the above discussion is expressed in terms of one dimensional integrals, the concept applies in any number of dimensions. For example, $\delta^{(3)}(\mathbf{r} - \mathbf{a})$ is a delta function in 3D space.

The reader does not have to go learn the theory of distributions in order to follow this appendix, but it is good to know that such a theory exists and justifies the manipulations done all the time with delta functions.

(a) Models for Delta Functions and two derivations of (2.1)

By "model" we mean a sequence of smooth functions which all have unit area and which in some limit become isolated about a certain point on the real axis. The essential requirements for a delta function candidate are these:

$$\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} dk\, \delta(k) = 1 \qquad \delta(k) = 0 \text{ for any } k>0 \text{ and for any } k<0 \tag{A.1}$$

These equations say that the area under $\delta(k)$ is 1 and the function $\delta(k)$ is isolated to an infinitely small neighborhood of $k = 0$. There are an infinite number of possible delta function models, and we shall consider several in this appendix, many of which are used in the main text.

Our first candidate delta function model is the pulse shown in (9.1) with $\tau/2$ set to $1/(2A)$,

$$\delta_1(k,A) \equiv A\,[\,\theta(k + \frac{1}{2A}) - \theta(k - \frac{1}{2A})\,] \tag{A.2}$$

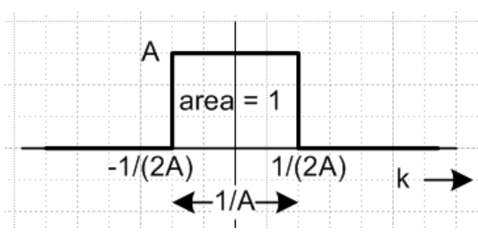


Fig A.1

In the limit $A\to\infty$, the box becomes very tall and very localized and maintains area 1, so

$$\lim_{A\to\infty} \delta_1(k,A) = \delta(k) . \tag{A.3}$$

Here are two candidates $\delta_2(k,A)$ and $\delta_{2'}(k,A)$ for which we just show engineering drawings and no equations,

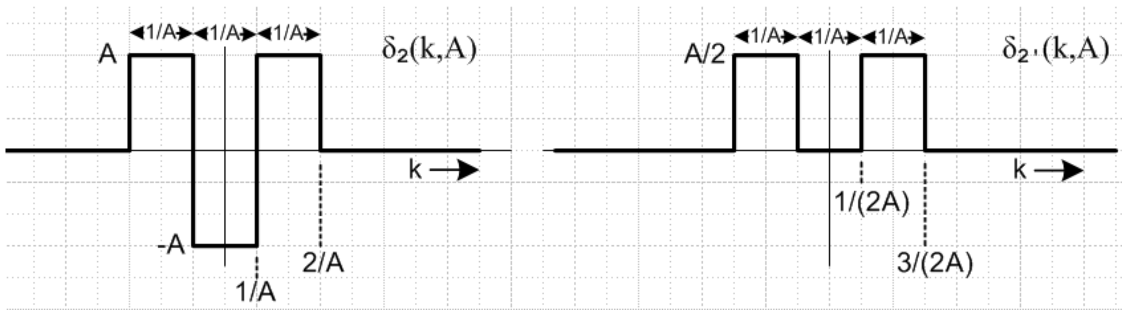


Fig A.2

Both these candidates meet our requirements (A.1). Notice that both have unit area, and both get horizontally compressed around $k = 0$ as $A \rightarrow \infty$ and both get "tall". Thus we can say

$$\lim_{A\rightarrow\infty} \delta_2(k,A) = \delta(k) \tag{A.4}$$

$$\lim_{A\rightarrow\infty} \delta_{2'}(k,A) = \delta(k) \quad . \tag{A.5}$$

For our first model δ_1 we get $\delta(0) = +\infty$ while for the last two models we get $\delta(0) = -\infty$ and $\delta(0) = 0$. It is in fact possible to construct a model in which $\delta(0)$ comes out being any desired real number. This number $\delta(0)$ has no significance because the point $k = 0$ is singular and the limit of $\delta(k)$ approaching this point from either direction does not exist. This is a much more serious matter than both limits existing and being different, which is the case for the Heaviside θ function which has a simple discontinuity at $t = 0$,

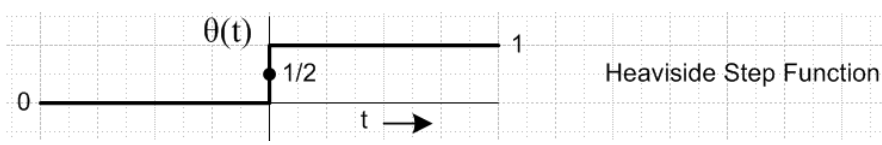


Fig A.3

From the left, the limit is 0, from the right, the limit is 1, and the Fourier-correct value at the discontinuous point is $\theta(0) = 1/2$ (as shown in Section 8 (c)).

Our next model of interest is this,

$$\delta_3(k,A) \equiv \frac{1}{\sqrt{4\pi A}} \exp(-k^2/4A) \tag{A.6}$$

which is a Gaussian centered at $k = 0$. This function has area = 1 for any $A > 0$,

```
int(exp(-k^2/(4*A))/sqrt(4*Pi*A), k = -infinity..infinity);
                                   1
```

The half-width of (A.6) occurs roughly when $k^2/4A = 1$, so the full width is then $\Delta k \approx 4\sqrt{A}$. Here is a plot with $A = .05$ for which $\Delta k \approx 4\sqrt{.05} = 0.9$,

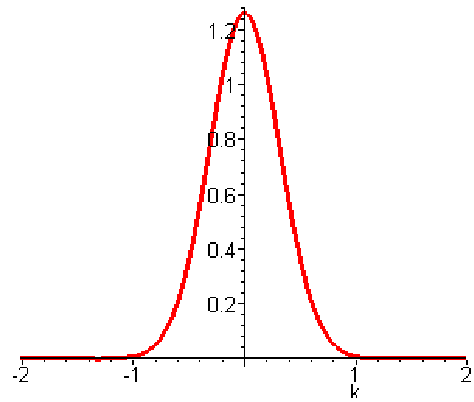


Fig A.4

As A→0, the Gaussian becomes taller and narrower and we then have

$$\lim_{A\to 0} \delta_3(k,A) = \lim_{A\to 0} \frac{1}{\sqrt{4\pi A}} \exp(-k^2/4A) = \delta(k). \tag{A.7}$$

Now consider this standard integral (GR 3.323.2 where $a = p^2$ and $b = q$),

$$\int_{-\infty}^{\infty} dx \, \exp[-(ax^2 + bx)] = \sqrt{\frac{\pi}{a}} \exp[(b^2/4a)] \quad . \tag{A.8}$$

Setting a = A, b = -ik gives

$$\int_{-\infty}^{\infty} dx \, e^{ikx} \exp(-Ax^2) = \sqrt{\frac{\pi}{A}} \exp(-k^2/4A) = 2\pi \left\{ \frac{1}{\sqrt{4\pi A}} \exp(-k^2/4A) \right\}$$

or

$$\int_{-\infty}^{\infty} dx \, e^{ikx} \exp(-Ax^2) = 2\pi \, \delta_3(k,A). \tag{A.9}$$

Taking the limit A→0 of both sides gives

$$\int_{-\infty}^{\infty} dx \, e^{ikx} = 2\pi\delta(k) \, . \tag{A.10}$$

This then is our first distribution theory derivation of (2.1).

Here is another candidate delta function model :

$$\delta_4(k,B) \equiv \frac{\sin(Bk)}{\pi k} = \frac{B}{\pi} \frac{\sin(Bk)}{Bk} = \frac{B}{\pi} \operatorname{sinc}(Bk) \tag{A.11}$$

The area under (B/π) sinc(Bk) is unity,

```
int((B/Pi)*sin(B*k)/(B*k), k = -infinity..infinity);
```

$$1$$

The half-width of (A.11) is determined roughly by the first zero of sin(Bk) = 0 so Bk = π. The full width of the peak is then $\Delta k = 2\pi/B$. Here is a plot for B = 10 with $\Delta k \approx 2\pi/10 = 0.6$.

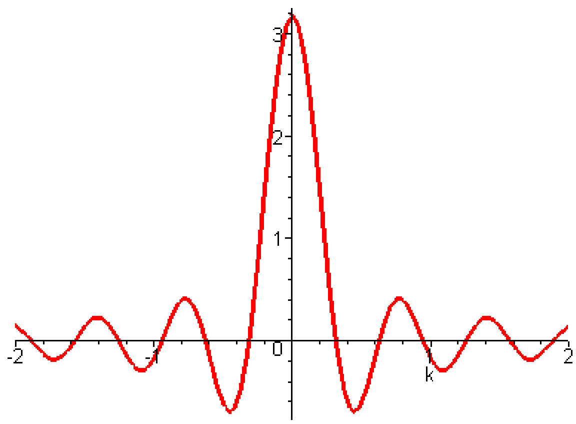


Fig A.5

As B gets large, the function shrinks in around k = 0, and we then have

$$\lim_{B\to\infty} \delta_4(k,B) = \lim_{B\to\infty} \frac{\sin(Bk)}{\pi k} = \lim_{B\to\infty} \frac{B}{\pi} \,\mathrm{sinc}(Bk) = \delta(k) \,. \qquad (A.12)$$

Now consider this simple integral

$$\int_{-B}^{B} dx\, e^{ikx} = 2\int_{0}^{B} dx \cos(kx) = 2\,\frac{\sin(kB)}{k} = 2\pi\,\frac{\sin(kB)}{\pi k} = 2\pi\ \delta_4(k,B) \qquad (A.13)$$

Taking the limit as B→0 we find

$$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\int_{0}^{\infty} dx \cos(kx) = 2\pi\,\delta(k) \qquad (A.14)$$

and we have a second derivation of (2.1).

(b) Models for Periodic Delta Functions

We start here with the following delta function model,

$$\delta_5(k,N) \equiv \frac{1}{2\pi}\,\frac{\sin[(N+1/2)k]}{\sin(k/2)} \qquad (A.15)$$

and we shall be interested in the limit N→∞. This particular candidate is periodic in k with period 2π, as we now show:

$\sin[(N+1/2)(k+2\pi)] = \sin[(N+1/2)k + 2\pi(N+1/2)] = \sin[(N+1/2)k + \pi] = -\sin[(N+1/2)k]$

$\sin[(k+2\pi)/2] = \sin(k/2 + \pi) = -\sin(k/2)$

$$\Rightarrow \frac{\sin[(N+1/2)(k+2\pi)]}{\sin[(k+2\pi)/2]} = \frac{\sin[(N+1/2)k]}{\sin(k/2)} \quad \Rightarrow \quad \delta_5(k+2\pi,N) = \delta_5(k,N) \qquad \text{(A.16)}$$

Looking at the peak at k = 0 for large N, the half width occurs at the first zero of $\sin[(N+1/2)k]$ so $Nk \approx \pi$. The full width is then $\Delta k \approx 2\pi/N$. Here is a plot of the central peak for N = 60 for which $\Delta k \approx 0.1$

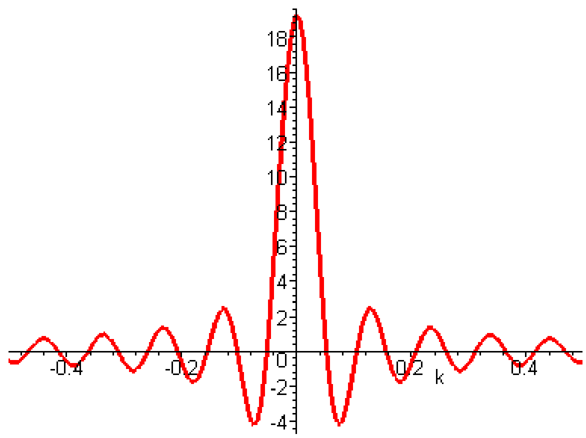


Fig A.6

The periodicity of δ_5 is apparent if we plot $2\pi\, \delta_5(k,N)$ for N = 60,

```
f := sin((N+1/2)*k)/sin(k/2);
```

$$f := \frac{\sin\left(\frac{121}{2}k\right)}{\sin\left(\frac{1}{2}k\right)}$$

```
N := 60:
plot(f, k = -10..10, numpoints = 1000);
```

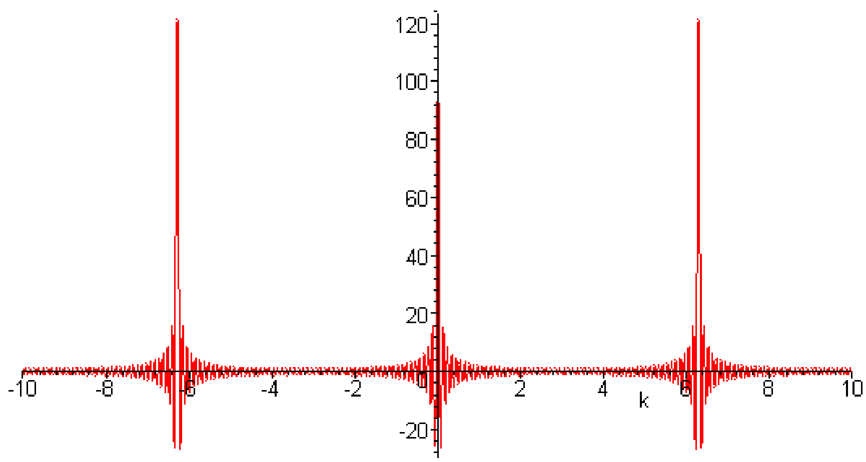


Fig A.7

where the peaks are separated by 2π. We are going to show that

$$\lim_{N\to\infty} \delta_5(k,N) = \delta(k) + \delta(k-2\pi) + \delta(k+2\pi) + \ldots = \sum_{m=-\infty}^{\infty} \delta(k - 2\pi m)\ . \qquad \text{(A.17)}$$

As N gets large, each red region of activity becomes isolated more and more to the location of the putative delta function peak. The function approaches 0 anywhere between the peaks in the following sense (which has a very distributional flavor). As $N \to \infty$, the numerator of $\delta_5(k,N)$ oscillates faster and faster. When averaged over any tiny region of width ε, we can make N large enough to make this average be arbitrarily small. For example, when averaged over 1 nanometer of the above plot near k = 2, we can find a sufficiently large N to make this average be smaller than 10^{-100}. This is the basic idea of something "washing out". Thus, we realize our delta function requirement that $\delta(k) = 0$ for $k \neq n2\pi$. The other requirement is that we must show that the area under each delta peak is unity.

Consider a close neighborhood of the central peak in Fig A.7. For large N, only a small region of k near the central peak $|k| < \Delta k \approx \pi/N$ contributes to δ_5 as the Figures above show, since sin[(N+1/2)k] oscillates so fast beyond this region. In this region we can approximate sin(k/2) by (k/2) so that

$$\delta_5(k,N) \equiv \frac{1}{2\pi}\frac{\sin[(N+1/2)k]}{\sin(k/2)} \approx \frac{1}{2\pi}\frac{\sin[(N+1/2)k]}{(k/2)} = \frac{(N+1/2)k}{2\pi(k/2)}\,\mathrm{sinc}[(N+1/2)k]$$

$$= \frac{(N+1/2)}{\pi}\,\mathrm{sinc}[(N+1/2)k] = \delta_4(k,N+1/2) \qquad \text{// using (A.11)}$$

We already know that the area under δ_4 is 1, and that it is a viable delta function model. But since δ_5 is periodic, *all* its peaks must look like δ_4. Broadening our scope to the entire k axis, we conclude that

$$\delta_5(k,N) \approx \sum_{m=-\infty}^{\infty} \delta_4(k - 2\pi m, N+1/2) \qquad \text{for large N} \qquad \text{(A.18)}$$

As $N \to \infty$ we then get

$$\lim_{N\to\infty} \delta_5(k,N) = \lim_{N\to\infty} \frac{1}{2\pi}\frac{\sin[(N+1/2)k]}{\sin(k/2)} = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad \text{(A.19)}$$

Our next multi-peak delta function candidate is the following,

$$\delta_6(k,N) \equiv \frac{1}{2\pi}\frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1}\frac{\sin^2[(N+1/2)k]}{2\pi\sin^2(k/2)}\ . \qquad \text{(A.20)}$$

This δ_6 has the same periodicity of δ_5 so has identical peaks spaced by 2π in k. Here is the central peak for N = 20

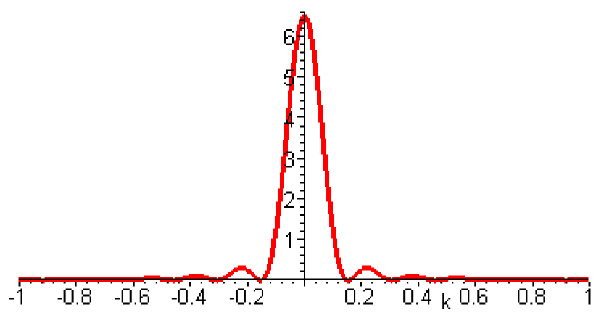


Fig A.8

and the width is $\Delta k \approx 2\pi/N$, the same as for δ_5. In the region of contribution, we again set $\sin(k/2) \approx k/2$ so that

$$\delta_6(k,N) \approx \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi\,(k/2)^2} = \frac{[(N+1/2)k]^2}{2\pi(2N+1)(k/2)^2} \operatorname{sinc}^2[(N+1/2)k] = \frac{(2N+1)}{2\pi} \operatorname{sinc}^2[(N+1/2)k]$$

$$= (B/\pi) \operatorname{sinc}^2(Bk) \qquad B = N+1/2$$

The area under δ_6 for any N is unity,

```
int( (B/Pi)*(sin(B*k)/(B*k))^2, k=-infinity..infinity);
                                        1
```

Since δ_6 has the same periodicity as δ_5, it has the same limit as $N\to\infty$

$$\lim_{N\to\infty} \delta_6(k,N) = \lim_{N\to\infty} \frac{\sin^2[(N+1/2)k]}{(2N+1)2\pi\,(k/2)^2} = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad \text{(A.21)}$$

Our interest in δ_6 is that it is a delta function model formed by *squaring* another delta function model.

Consider next the following candidate delta function model,

$$\delta_7(k, a) \equiv (1/\pi) \frac{(1-a^2)\,[\cos^2(k)]}{1+a^2+2a\cos(2k)} \qquad \text{(A.22)}$$

We are interested in this model as $a \to -1$. Here is a motivating plot of δ_7 for $a = -0.9$:

```
d7 := (1/Pi)*(1-a^2)*cos(k)^2 / ( 1+a^2+2*a*cos(2*k) );
```

$$d7 := \frac{(1-a^2)\cos(k)^2}{\pi\,(1+a^2+2\,a\cos(2\,k))}$$

```
a := -0.9: plot(d7, k = -4..4);
```

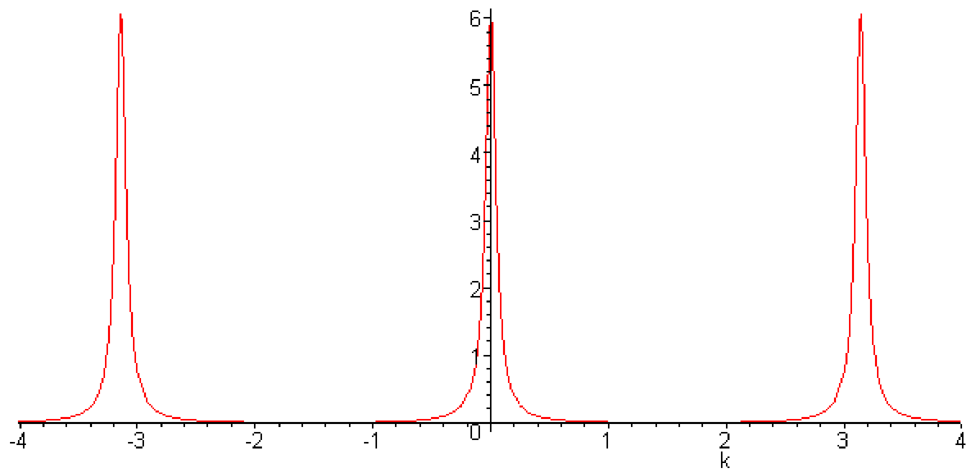


Fig A.9

First of all, we can see that δ_7 is periodic in k having period π, since both $\cos^2(k)$ and $\cos(2k)$ have period π. Near a = -1, the denominator approaches $2 - 2\cos(2k) = 2(1-\cos(2k)) = 4\sin^2(k)$ which vanishes at $k = m\pi$ for m integer. As long as we avoid these points, we can see that $\lim_{a\to-1} \delta_7(k, a) = 0$ for $k \neq m\pi$ due to the $(1-a^2)$ factor in the numerator. We need only show that the integral if δ_7 is 1 when taken in a small region around one of the delta peaks. As before we consider the peak at k = 0 and compute the integral

$$\lim_{\varepsilon\to0} \int_{-\varepsilon}^{\varepsilon} dk\, [\lim_{a\to-1}\delta_7(k, a)] = \lim_{\varepsilon\to0} \lim_{a\to-1} \int_{-\varepsilon}^{\varepsilon} dk\ \{(1/\pi)\ \frac{(1-a^2)\,[\cos^2(k)]}{1+a^2+2a\cos(2k)}\}$$

$$= \lim_{a\to-1} \lim_{\varepsilon\to0} \int_{-\varepsilon}^{\varepsilon} dk\ \{(1/\pi)\ \frac{(1-a^2)\,[\cos^2(k)]}{1+a^2+2a\cos(2k)}\} \qquad \text{// interchange limit order}$$

$$= \lim_{a\to-1} \lim_{\varepsilon\to0}\{(1-a^2)\ \int_{-\varepsilon}^{\varepsilon} dk\, \frac{1}{1+a^2+2a\cos(2k)}\} \qquad \text{// } \cos(k) \approx 1 \text{ for } \varepsilon << 1$$

$$= (1/\pi) \lim_{\varepsilon\to0} \lim_{a\to-1}\ \{(1-a^2)\ \int_{-\varepsilon}^{\varepsilon} dk\, \frac{1}{1+a^2+2a\cos(2k)}\} \qquad \text{// interchanged limit order again}$$

In passing, note that the $\cos^2(k)$ numerator factor really plays no role in things and we could have set it to 1 in δ_7, but we kept it since it appears there in our AMI application. We now have Maple do the integral as follows,

```
assume(a<0,a>-1);
Int(1/(1+a^2+2*a*cos(2*k)),k=-epsilon..epsilon);
```

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{1+a^2+2\,a\cos(2\,k)}\,dk$$

```
int(1/(1+a^2+2*a*cos(2*k)),k=-epsilon..epsilon): simplify(%)
```

$$2\,\frac{\arctan\left(\frac{(a-1)\tan(\varepsilon)}{a+1}\right)}{a^2-1}$$

We now continue the above evaluation,

$$= (1/\pi)\ \lim_{\varepsilon\to 0}\ \lim_{a\to -1}\{(1-a^2)\ [\ 2\,\frac{1}{a^2-1}\ \tan^{-1}(\ \frac{a-1}{a+1}\ \tan\varepsilon\ )\ ]$$

$$= -(2/\pi)\ \lim_{\varepsilon\to 0}\ \lim_{a\to -1}\{\ \tan^{-1}(\ \frac{a-1}{a+1}\ \tan\varepsilon\ )\ \}$$

$$= -\ (2/\pi)\ \lim_{\varepsilon\to 0}\ \{\ \tan^{-1}(-\ \infty)\ \} \qquad // \text{ argument of } \tan^{-1} \text{ is } (\frac{-1-1}{+0}\tan\varepsilon) = (-\infty)$$

$$= -\ (2/\pi)\ \lim_{\varepsilon\to 0}\ \{\ -\pi/2\ \}\ =\ -\ (2/\pi)(-\pi/2)\ =\ 1\ .$$

In this model, the area under δ_7 is not unity for any value of a, so we have to deal with both limits in our evaluation. There are Moore-Osgood theorem subtleties involving the limit order interchanges which could be reviewed, but we omit that level of detail.

Therefore we have shown that in the region of the central peak.

$$\lim_{a\to -1}\delta_7(k,a)\ = \delta(k) \qquad\qquad -\pi < k < \pi$$

Since δ_7 is periodic with period π, we know that the full result for all k is this

$$\lim_{a\to -1}\delta_7(k,a)\ =\ \lim_{a\to -1}\ (1/\pi)\ \frac{(1-a^2)\ [\cos^2(k)]}{1+a^2+2a\cos(2k)}\ =\ \sum_{m=-\infty}^{\infty}\delta(k-m\pi) \qquad \text{(A.23a)}$$

If we remove $[\cos^2(k)]$ from the limit and then divide both sides of the right equation above by this value, we get on the right that $[\cos^2(k)] = \cos^2(m\pi) = 1$, so the following is also true

$$\lim_{a\to -1}\ (1/\pi)\ \frac{(1-a^2)}{1+a^2+2a\cos(2k)}\ =\ \sum_{m=-\infty}^{\infty}\delta(k-m\pi) \qquad \text{(A.23b)}$$

which is really a more fundamental result. Changing $a\to -a$ this can also be written

$$\lim_{a\to +1}\ (1/\pi)\ \frac{(1-a^2)}{1+a^2-2a\cos(2k)}\ =\ \sum_{m=-\infty}^{\infty}\delta(k-m\pi) \qquad \text{(A.23c)}$$

Going back to (A.23a), if we change from k to k' = k + π/2, then

$$\cos(k) = \cos(k'-\pi/2) = \sin(k')$$
$$\cos(2k) = \cos(2k'-\pi) = -\cos(2k') \ .$$

Then δ_7 may be written

$$\delta_7(k, a) \equiv (1/\pi)\ \frac{(1-a^2)\,[\cos^2(k)]}{1+a^2+2a\cos(2k)} \quad => \quad \delta_7(k'-\pi/2, a) \equiv (1/\pi)\ \frac{(1-a^2)\,[\sin^2(k')]}{1+a^2-2a\cos(2k')}$$

Then from (A.23a)

$$\lim_{a\to -1}\delta_7(k'-\pi/2, a) = \sum_{m=-\infty}^{\infty} \delta(k'-\pi/2 - m\pi) = \sum_{m=-\infty}^{\infty} \delta(k'- [m+1/2]\pi) \ .$$

Change the summation index to n = 2m+1, so that m+1/2 = n/2. The n sum then includes only odd integers from -∞ to ∞, so

$$\lim_{a\to -1}\delta_7(k'-\pi/2, a) = \sum_{n=\pm\text{odd}} \delta(k'-n\pi/2) \ .$$

Thus we have arrived at a new multi-delta function model $\delta_8(k',a) \equiv \delta_7(k'-\pi/2, a)$ to get

$$\delta_8(k, a) \equiv (1/\pi)\ \frac{(1-a^2)\,[\sin^2(k)]}{1+a^2-2a\cos(2k)} \tag{A.24}$$

$$\lim_{a\to -1}\delta_8(k, a) = \lim_{a\to -1} (1/\pi)\ \frac{(1-a^2)\,[\sin^2(k)]}{1+a^2-2a\cos(2k)} = \sum_{m=\pm\text{odd}} \delta(k-m\pi/2) \tag{A.25a}$$

This is the delta model needed to take the p→ 1 limit of our AMI spectrum in Section 37.

If we extract $[\sin^2(k)]$ from the limit and divide both sides of the rightmost equation by this quantity and note that $\sin^2(m\pi/2) = 1$ for all odd values of m, we get these alternate forms:

$$\lim_{a\to -1} (1/\pi)\ \frac{(1-a^2)}{1+a^2-2a\cos(2k)} = \sum_{m=\pm\text{odd}} \delta(k-m\pi/2) \tag{A.25b}$$

$$\lim_{a\to +1} (1/\pi)\ \frac{(1-a^2)}{1+a^2+2a\cos(2k)} = \sum_{m=\pm\text{odd}} \delta(k-m\pi/2) \tag{A.25c}$$

(c) Derivation of (13.2) and (13.3)

The first goal here is to derive this equation,

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \quad . \qquad (13.2)$$

Consider this finite version of the sum appearing on the left side of (13.2). We add and subtract 1 to obtain the sum as the these three terms,

$$\sum_{n=-N}^{N} e^{ink} = (1 + e^{ik} + e^{i2k} + ... + e^{iNk}) + (1 + e^{-ik} + e^{-i2k} + ... + e^{-iNk}) - 1 \quad . \qquad (A.26)$$

The following is the standard formula for summing N terms of a geometric series,

$$1 + x + x^2 + ... + x^N = (1 - x^{N+1})/(1-x), \qquad (A.27)$$

Using this formula first for $x = e^{ik}$ and then for $x = e^{-ik}$, we may write (A.26) as

$$\sum_{n=-N}^{N} e^{ink} = (1 - e^{ik(N+1)})/(1-e^{ik}) + (1 - e^{-ik(N+1)})/(1-e^{-ik}) - 1 \qquad (A.28)$$

In the first term multiply top and bottom by $e^{-ik/2}$ to get

$$\text{first term} = (e^{-ik/2} - e^{ik(N+1/2)})/ (e^{-ik/2}- e^{ik/2}) = (e^{-ik/2} - e^{ik(N+1/2)}) / [-2i\sin(k/2)] \quad .$$

Since the second term is the complex conjugate of the first, we get

$$\text{second term} = (e^{ik/2} - e^{-ik(N+1/2)}) / [2i\sin(k/2)]$$

Therefore

$$\sum_{n=-N}^{N} e^{ink} = \text{second term} + \text{first term} - 1$$

$$= [e^{ik/2} - e^{-ik(N+1/2)} - e^{-ik/2} + e^{ik(N+1/2)}] / [2i\sin(k/2)] - 1$$

$$= [2i \sin(k/2) + 2i\sin[k(N+1)/2]] / [2i\sin(k/2)] - 1$$

$$= [\sin(k/2) + \sin[k(N+1)/2]] / [\sin(k/2)] - 1$$

$$= \sin[k(N+1)/2] / \sin(k/2)$$ // see also Gradshteyn and Ryzhik, p 37, 1.342.2

and we arrive at this result (valid for any positive integer N):

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \tag{A.29}$$

From (A.15) we recognize {...} in (A.29) as $\delta_5(k,N)$, so (A.29) says

$$\sum_{n=-N}^{N} e^{ink} = 2\pi\, \delta_5(k,N) \quad . \tag{A.30}$$

We have thus derived equation (13.3).

We then take the limit N→∞ of both sides of this equation. The right side is given by (A.19) so we get

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \tag{A.31}$$

and we have derived (13.2) as promised. Both sides are visibly periodic with period 2π.

(d) Undoing the limit N→ ∞ : the meaning of δ(0)

Go back now to (A.29) and evaluate both sides at k = 0:

$$\sum_{n=-N}^{N} 1 = 2\pi \lim_{k\to 0}\left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} = \lim_{k\to 0}\left\{ \frac{\sin[(N+1/2)k]}{(k/2)} \right\} \tag{A.32}$$

$$= (2N+1) \lim_{k\to 0} \mathrm{sinc}[\,(N+1/2)k] = (2N+1).$$

But this is exactly the sum shown on the left, so we find that our k = 0 limit of (A.29) happily says

$$(2N+1) = (2N+1) .$$

Taking the limit N→ ∞ of (A.30) and then the limit k → 0 we find that

$$\sum_{n=-\infty}^{\infty} 1 = [\, 2\pi\delta(0)] \quad . \tag{A.33}$$

We noted earlier that δ(0) is model dependent and for this model δ_5 we get δ(0) = +∞. There are times when we want to "undo the limit" N→∞, to get

$$\sum_{n=-N}^{N} 1 = [\, 2\pi\delta(0)]_{\mathrm{undone}} = (2N+1) . \tag{A.34}$$

In this "undoing" we have to be careful not to use the delta function property $\delta(x/a) = a\,\delta(x)$ since this is not valid in the "pre-limit".

Example 1

When we deal with pulse trains, we will be adding exponentials of the form $\exp(in\omega T_1)$. From (A.31) we therefore have,

$$\sum_{n=-\infty}^{\infty} e^{in\omega T_1} = \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ . \tag{A.35}$$

If we evaluate the above formula at $\omega = 0$, we get

$$\sum_{n=-\infty}^{\infty} 1 \;=\; \sum_{m=-\infty}^{\infty} 2\pi\delta(-\,2\pi m)$$

We now observe that $\delta(-2\pi m) = \delta_{m,0}\,\delta(0)$, so we then get, as in (A.33),

$$\sum_{n=-\infty}^{\infty} 1 \;=[\,2\pi\,\delta(0)] \tag{A.36}$$

We understand this as a symbolic limit. We can now undo the limit by replacing the sum endpoints with -N and N, and replace $2\pi\delta(0)$ with (2N+1), and the result is consistent. Had we rescaled the delta function by saying for example $\delta(-2\pi m) = (1/2\pi)\,\delta(-m)$, we would end up with a contradiction when we tried to "undo the limit".

Example 2

Consider the *square* of the sum shown in Example 1,

$$\{\sum_{n=-\infty}^{\infty} e^{in\omega T_1}\}^2 \;=\{\sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ \}^2 \tag{A.37}$$

We write the RHS using m and k for our two summation indices, then we move both summations to the left. Inside this double sum we get

$$\delta(\omega T_1 - 2\pi m)\delta(\omega T_1 - 2\pi k)\ \ .$$

Since the first delta function will "pin" ωT_1 to the values $2\pi m$, we can replace ωT_1 with $2\pi m$ in the second delta function and not change a thing, obtaining

$$\delta(\omega T_1 - 2\pi m)\delta(2\pi m - 2\pi k) = \delta(\omega T_1 - 2\pi m)\ \ \delta_{m,n}\,\delta(0)\quad . \tag{A.38}$$

The Kronecker delta $\delta_{m,n}$ now removes one of the summations, and we end up with this result:

$$\{ \sum_{n=-\infty}^{\infty} e^{in\omega T_1} \}^2 = [\, 2\pi\delta(0)\,]\, \{ \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \quad . \tag{A.39}$$

Evaluating the above at $\omega = 0$ yields

$$\{ \sum_{n=-\infty}^{\infty} 1 \}^2 = [\, 2\pi\delta(0)\,]\ [\, 2\pi\delta(0)\,] \quad . \tag{A.40}$$

And if we now undo the limit, we get this self-consistent result,

$$\{ \sum_{n=-N}^{N} 1 \}^2 = [\, 2\pi\delta(0)\,]\ [\, 2\pi\delta(0)\,] = (2N+1)^2 . \tag{A.41}$$

Once again, $\delta(0)$ is really undefined and model dependent, but for our δ_5 model we use $2\pi\delta(0)$ just as a shorthand for the limit of (2N+1) as $N \rightarrow \infty$. The number (2N+1) will be the number of pulses in a pulse train and that pulse train becomes infinitely long as $N \rightarrow \infty$. In such a limit, quantities like average energy per pulse remain finite.

(e) The function $\Theta(a \le x \le b)$ and related sums

In equation (2.2) we noted that [$\theta(x)$ is the Heaviside step function of Fig 1, sometimes written as H(x)]

$$\int_a^b dx\ \delta(x-y)f(x) = f(y)\theta(b-y)\theta(y-a) \qquad a < b \tag{2.2}$$

which we write here as

$$\int_a^b dx'\ \delta(x'-x)f(x') = f(x)\theta(b-x)\theta(x-a) \qquad a < b \tag{A.42}$$

The theta functions just set the result to 0 if the delta function hit lies outside the interval (a,b).

Function $\theta(x)$ is the Heaviside step function having these properties:

$$\theta(x) = \begin{cases} 1 & \text{if } x>0 \\ 1/2 & \text{if } x=0 \\ 0 & \text{if } x<1 \end{cases} \quad . \tag{A.43}$$

These theta functions are always a bit confusing, and sometimes things are clearer using a different notation. Suppose we define the following new function,

$$\Theta(a \le x \le b) \equiv \Theta(a,x,b) \equiv \begin{cases} 1 & \text{if } a<x<b \\ 1/2 & \text{if } x=a \text{ or } x=b \\ 0 & \text{if } x<a \text{ or } x>b \end{cases} \qquad a < b \qquad (A.44)$$

The notation $\Theta(a \le x \le b)$ is merely a suggestive way to write the function $\Theta(a,x,b)$.

We shall now state and prove a few simple "theorem" regarding this function Θ.

Fact: $\Theta(a \le x \le b) = \theta(b-x)\theta(x-a) \qquad a < b \qquad$ (A.45)

Proof. We just exhaust all cases:

$x < a$: $\theta(b-x)\theta(x-a) = \theta(b-x) * 0 = 0$
$x = a$: $\theta(b-x)\theta(x-a) = \theta(b-a)\theta(a-a) = 1 * 1/2 = 1/2$
$a < x < b$: $\theta(b-x)\theta(x-a) = 1*1 = 1$
$x=b$: $\theta(b-x)\theta(x-a) = \theta(b-b)\theta(b-a) = 1/2 * 1 = 1/2$
$x > b$: $\theta(b-x)\theta(x-a) = 0* \theta(x-a) = 0$

Therefore, we may write (A.42) in this more friendly manner,

$$\int_a^b dx'\ \delta(x'-x)f(x') = f(x)\ \Theta(a \le x \le b) \qquad a < b \qquad (A.46)$$

Fact: $\Theta(a \le x+c \le b) = \Theta(a-c \le x \le b-c)$ (A.47)

Proof: The inequality notation makes this fact seem completely obvious, but we will just make sure by writing out both sides of the equation:

$$\Theta(a \le x+c \le b) \equiv \Theta(a,x+c,b) \equiv \begin{cases} 1 & \text{if } a< x+c <b \\ 1/2 & \text{if } x+c =a \text{ or } x+c =b \\ 0 & \text{if } x+c <a \text{ or } x+c >b \end{cases} \qquad a < b$$

$$\Theta(a-c \le x \le b-c) \equiv \Theta(a-c,x,b-c) \equiv \begin{cases} 1 & \text{if } a-c <x<b-c \\ 1/2 & \text{if } x= a-c \text{ or } x= b-c \\ 0 & \text{if } x<a-c \text{ or } x>b-c \end{cases} \qquad a-c < b-c$$

The next fact is not quite so obvious but is very useful:

Fact: $\displaystyle\sum_{m=-\infty}^{\infty} \Theta(m\alpha-\alpha/2 \le x \le m\alpha+\alpha/2) = 1 \qquad \alpha > 0 \qquad -\infty < x < \infty \qquad$ (A.48)

Proof: Let's write out some of the terms in this sum. Here we show terms for m = -1, 0, and 1

$$.... + \Theta(-\alpha-\alpha/2 \le x \le -\alpha+\alpha/2) + \Theta(0-\alpha/2 \le x \le 0+\alpha/2) + \Theta(\alpha-\alpha/2 \le x \le \alpha+\alpha/2) + ...$$

or

$$.... + \Theta(-(3/2)\alpha \le x \le -(1/2)\alpha) + \Theta(-(1/2)\alpha \le x \le (1/2)\alpha) + \Theta((1/2)\alpha \le x \le (3/2)\alpha) + ...$$

The terms of the m sum therefore *partition* the real x axis into intervals of width α. If x falls within one of these intervals, then only the single term covering that interval contributes in the m sum and the sum is then 1. If x happens to fall exactly on the boundary between two intervals, then each of those intervals contributes 1/2 to the sum, and the sum is again 1. For example, if $x = (1/2)\alpha$, then each of the rightmost two terms shown above contributes 1/2. Therefore the sum is 1 for all possible values of x.

Fact: $$\sum_{m=-\infty}^{\infty} \Theta(-\alpha/2 \le x\text{-}m\alpha \le +\alpha/2) = 1 \qquad \alpha > 0 \qquad -\infty < x < \infty \qquad \text{(A.49)}$$

<u>Proof</u>: Apply (A.47) to the Θ function shown in the sum :

$$\Theta(a \le x+c \le b) = \Theta(a-c \le x \le b-c) \qquad a = -\alpha/2, \quad c = -m\alpha, \quad b = \alpha/2$$

$$\Theta(-\alpha/2 \le x\text{-}m\alpha \le \alpha/2) = \Theta(-\alpha/2+m\alpha \le x \le \alpha/2+m\alpha)$$

Therefore, summing both sides and using (A.48),

$$\sum_{m=-\infty}^{\infty} \Theta(-\alpha/2 \le x\text{-}m\alpha \le \alpha/2) = \sum_{m=-\infty}^{\infty} \Theta(-\alpha/2+m\alpha \le x \le \alpha/2+m\alpha) = 1$$

Corollary: $$\sum_{m=-\infty}^{\infty} \Theta(-\alpha/2 \le x + m\alpha \le +\alpha/2) = 1 \qquad \alpha > 0 \qquad -\infty < x < \infty \qquad \text{(A.50)}$$

<u>Proof:</u> For any summand f(m) it is clear that $\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} f(-m)$, so (A.50) is the same as (A.49).

(f) The product of two delta functions and more on δ(0)

This subsection is a sort of coda on the subject of δ(0) where we further attempt to justify the use of δ(0) even though δ(0) is formally undefined. We continue the distribution discussion begun at the start of this Appendix.

Consider a voltage pulse v(t) with a shape corresponding to one our delta function models. If this voltage is placed across a resistor of value R = 1, the energy in the pulse is given by

$$E = \int_{-\infty}^{\infty} dt\, v^2(t) \quad . \qquad\qquad \text{energy = time integral of power}$$

If we take the limit v(t)→δ(t), what happens? One is tempted to say

$$E = \int_{-\infty}^{\infty} dt\, \delta^2(t) = \int_{-\infty}^{\infty} dt\, \delta(t)\, \delta(t) = \delta(0) \int_{-\infty}^{\infty} dt\, \delta(t) = \delta(0) * 1 = \delta(0) = +\infty$$

and one concludes (correctly) that the energy in a delta function pulse is infinite and positive. There are several problems with the this analysis.

First, we have already seen that with different δ models, we can get δ(0) to be any number we want, including +∞. -∞ and 0. Very embarrassing.

Second, the distribution $<\delta_\xi{}^2, \varphi> = \delta(\xi)\varphi(\xi)$ is not a sensible distribution since this linear functional maps into a real number which is undefined at ξ = 0. In general the product of two distributions (in the sense we use it here) is not even defined in the realm of distribution theory, unless one or both distributions are regular functions. In that case we could have, for example,

$$< f\,\delta_\xi, \varphi > = < \delta_\xi, f\varphi> = \int_{-\infty}^{\infty} dx\ \delta(x-\xi)f(x)\varphi(x) = f(\xi)\varphi(\xi) = \text{ well defined }\ .$$

Some people have tried to incorporate products of singular distributions into distribution theory, but it is not clear how their results apply in our current context (see for example Colombeau 1990).

Note that there are other meanings of the product of two distributions. One is called a convolution product which is like f(g(x)) for functions, while the other involves multiple variables like $\delta^{(2)}(\mathbf{r}\text{-}\mathbf{r}') = \delta(x\text{-}x')\delta(y\text{-}y')$. Neither of these products involves products of symbolic functions in the same variable such as δ(t)δ(t).

So, how might we compute the energy in a delta function voltage pulse? The only reasonable thing to do is to back δ(t) off to one of its models and see what happens. For example, suppose we take $\delta_1(t,A)$ as stated in (A.2), which is the simple box model of height A and width 1/A. Then

$$E = \lim_{A\to\infty} \{ \int_{-\infty}^{\infty} dt\ \ [\delta_1(t,A)]^2\ \ \}$$

Now $[\delta_1(t,A)]^2$ is a box of width 1/A and height A^2 so its area is A. Then

$$E = \lim_{A\to\infty} \{ A \} = +\infty \qquad\qquad // \text{ recall } \delta(0) = +\infty$$

Suppose we take our deviant delta function model (A.4) $\delta_2(k,A)$ shown in Fig A.2 left. We get

$$E = \lim_{A\to\infty} \{ \int_{-\infty}^{\infty} dt\ \ [\delta_2(t,A)]^2\ \ \}$$

Since the pulse is squared, the area under $[\delta_2(t,A)]^2$ is 3A and we then get

$$E = \lim_{A\to\infty} \{ 3A \} = +\infty \qquad\qquad // \text{ recall } \delta(0) = -\infty$$

Finally, for the our second deviant model (A.5) $\delta_{2'}(k,A)$ shown in Fig A.2 right we get

$$E = \lim_{A\to\infty}\{ A/2 \} = +\infty \qquad\qquad // \text{ recall } \delta(0) = 0$$

Thus, we get E = +∞ regardless of the value of δ(0) in the model.

Despite this discussion which shows that $\delta(0)$ is undefined, we nevertheless use $\delta(0)$ with a particular model in mind because it allows us to avoid dealing with specific boundaries in equations.

In Example 2 of the previous subsection, we saw the use of $2\pi\delta(0)$ as meaning 2N+1 for a very long pulse train starting at -N in the distant past and ending at +N in far future. We would rather think in terms of an infinite pulse train and $2\pi\delta(0)$, but the physical meaning is a very long pulse train and 2N+1. It is just a convenient notation.

Another common example has to do with "box normalization" which in one dimension is the model presented as (A.13),

$$\int_{-L/2}^{L/2} dx\, e^{ikx} = 2\pi \frac{\sin(kL/2)}{\pi k} = 2\pi\ \delta_4(k,L/2)$$

$$\int_{-L/2}^{L/2} dx\, e^{i0x} = L = 2\pi\delta(0)\ .$$

In this case, we use $2\pi\delta(0)$ to represent the length of a box which is some very large L. Rather than carry the large but finite L along in all equations, we can use $2\pi\delta(0)$ as needed represent this long length.

The conclusion here is that we can use $2\pi\delta(0)$ as a notational device provided we are very careful as to the meaning of that device. We must always know how to undo the limit, and that implies a specific delta function model for a specific application.

Appendix B: Derivation of a Certain Identity

Theorem: For N and s both integers (N > 0)

$$\sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = N \sum_{m=-\infty}^{\infty} \delta_{s,mN} \quad . \tag{B.1}$$

This identity is used in Section 27 (b) on the Discrete Fourier Transform.

To prove this relation, we shall first show that each side is periodic in s with period N, then we shall verify that the relation is true for s = 0,1,2...N-1. We will have then shown that the relation is true for all integer values of s.

First, give names to the two sides of (B.1)

$$f(s) \equiv \sum_{m=0}^{N-1} e^{+ims(2\pi/N)} \qquad\qquad g(s) \equiv N \sum_{m=-\infty}^{\infty} \delta_{s,mN} \,.$$

Function f(s) is periodic as claimed because

$$f(s+kN) = \sum_{m=0}^{N-1} e^{+im(s+kN)(2\pi/N)} = \sum_{m=0}^{N-1} e^{+ims(2\pi/N)} e^{imkN(2\pi/N)} = \sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = f(s)$$

Function g(s) is periodic as claimed because (m' ≡ m-k)

$$g(s+kN) \equiv N \sum_{m=-\infty}^{\infty} \delta_{s+kN,mN} = N \sum_{m=-\infty}^{\infty} \delta_{s,(m-k)N} = N \sum_{m'=-\infty}^{\infty} \delta_{s,(m')N} = g(s)$$

Thus, each side of (B.1) is periodic in s with period N.

Now, consider the proposed equality:

$$\sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = N \sum_{m=-\infty}^{\infty} \delta_{s,mN} \quad . \tag{B.1}$$

For s = 0, the above claims

$$\sum_{m=0}^{N-1} 1 = N \sum_{m=-\infty}^{\infty} \delta_{0,mN} = N \ \delta_{0,0} = N \,.$$

But this is true since the left side sum is obviously N as well. Thus, (B.4) is valid for s = 0.

For s = 1,2,3....N-1, we *claim* that on the left side of (B.1) we are adding N equally spaced points around a circle in the complex phasor plane, and therefore the sum on the left side is zero. Meanwhile, the right side is *also* zero because for any of these s values, there is no integer m such that s = mN hence $\delta_{s,mN} = 0$. Thus, if one accepts the circle argument, one finds that for all these s values both sides of (B.1) vanish.

To avoid the circle construction, we can simply compute the left side of (B.1) using the formula

$$\sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = 1 + x + x^2 + ... + x^{N-1} = (x^N - 1)/(x-1) \qquad \text{with } x = e^{+is(2\pi/N)}$$

For s = 1,2,3....N-1 the phasor $x = e^{+ims(2\pi/N)} \neq 1$, so denominator $(x-1) \neq 0$. Meanwhile,

$$x^N = e^{+is(2\pi/N)N} = e^{+is2\pi} = 1 \qquad \text{for any integer s}$$

Therefore numerator $(x^N - 1) = 0$ and the sum thus vanishes for these values of s.

Thus we have shown that (B.1) is valid for s = 0,1,2...N-1, and since both sides of (B.1) are periodic in s with period N, it must be that (B.1) is valid for all integers s.

Appendix C: The Fourier Transform and its relation to the Hilbert Transform

In this Appendix we refer to the Fourier Integral Transform simply as the Fourier transform. We develop more "facts" about Fourier transforms, including new notations, and present a set of closely related examples. The pf pseudofunction and principal part integrals are introduced in the context of what we call "the pole avoidance rule". The connection between the Fourier and Hilbert transforms is then used as an exercise in applying the developed methods.

(a) Fourier Transform Notations

Recall from Section 1 the statement of the Fourier transform, derived in Section 2,

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform} \qquad (1.2)$$

We used lower case for a function of time like x(t), and upper case for the spectral components X(ω). Although convenient in many situations, this notation is a bit limiting for more general use, so we replace X(ω) with x^(ω). The above can then be written as,

$$x\hat{}(\omega) = k \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform}$$

$$x(t) = (1/2\pi k) \int_{-\infty}^{\infty} d\omega\, x\hat{}(\omega)\, e^{+i\omega t} \qquad \text{expansion = inverse transform}$$

Here we have added an arbitrary constant k to allow for other scalings of the Fourier Transform. The projection has k, the inversion has k^{-1} as shown. For us, k = 1, but other sources might have $k = (2\pi)^{-1}$ or perhaps $k = 1/\sqrt{2\pi}$ to make the two equations symmetric.

The first line defines the Fourier transform of some arbitrary function x(t), but the second line acts only on a function x^(ω) which is already a Fourier transform. We would like to have the second line act on an arbitrary function as well. To do this, we replace x^(ω) by f(ω) and treat the second line as an operation one applies to some arbitrary function f(ω),

$$f\hat{}^{-1}(t) = (1/2\pi k) \int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{+i\omega t} \qquad \text{inverse transform}$$

This then defines an operation performed on f(ω) to generate $f\hat{}^{-1}(t)$ which is, by definition, the *inverse* Fourier transform of f(ω). So changing the dummy integration variable names both to u we get

$$f\hat{}(\omega) = k \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} \qquad \text{Fourier transform of f(u)}$$

$$f\hat{}^{-1}(t) = (1/2\pi k) \int_{-\infty}^{\infty} du\, f(u)\, e^{+iut} \qquad \text{inverse Fourier transform of f(u)} \qquad (C.1)$$

We think of both these equations as defining certain operations on an arbitrary function f(u). The function f(u) must be in the class of functions described in Section 1(b) in order that the integral converge to a function, though the class rules may be violated if one allows the transform and/or its inverse to be a distribution.

From the first line of (C.1) we find that

$$f\hat{}(-\omega) = k \int_{-\infty}^{\infty} du\, f(u)\, e^{+i\omega u} = k \int_{-\infty}^{\infty} du\, f(-u)\, e^{-i\omega u} = [f(-u)]\hat{}(\omega)$$

from which we obtain

Fact 0: $[f(-u)]\hat{}(-\omega) = f\hat{}(\omega)$ (C.2)

From the second line of (C.1) we find

$$f\hat{}^{-1}(-\omega) = (1/2\pi k) \int_{-\infty}^{\infty} du\, f(u)\, e^{-iu\omega} = (1/2\pi k^2)\; k \int_{-\infty}^{\infty} du\, f(u)\, e^{-iu\omega} = (1/2\pi k^2)\, f\hat{}(\omega) \quad .$$

We are dealing here with three distinct functions: f(u), $f\hat{}(u)$ and $f\hat{}^{-1}(u)$, but we just showed that

Fact 1: $f\hat{}^{-1}(u) = (1/2\pi k^2)\, f\hat{}(-u)$ (C.3)

so two of these three functions have a simple relationship.

Notice in Fact 1 that one cannot simply suppress the argument u on both sides since u appears on the left and –u appears on the right. When an argument is the same on both sides of an equation, or when an argument is not needed, it can be suppressed to reduce clutter.

The fact that the Fourier transform is valid means that the inverse Fourier transform of the Fourier transform of a function *is* that function. Similarly, the Fourier transform of the inverse Fourier transform of a function *is* that function. This was clear in (1.1) and (1.2) and in the new notation this becomes

Fact 2: $[f\hat{}(\omega)]\hat{}^{-1}(t) = [f\hat{}^{-1}(\omega)]\hat{}(t) = f(t)$

or

$[f\hat{}]\hat{}^{-1} = [f\hat{}^{-1}]\hat{} = f$ (C.4)

This provides an example of suppressing arguments to declutter a simple equation. Notice that the constant k does not appear in Fact 2. Consider then this statement of Fact 2,

$$(f\hat{})\hat{}^{-1}(t) = f(t)$$

Fact 1 applied to $f \rightarrow f\hat{}$ says

$$(f\hat{})\hat{}^{-1}(t) = (1/2\pi k^2)\, (f\hat{})\hat{}(-t)$$

so that

$(1/2\pi k^2)$ (f^)^(-t) = f(t)

or

(f^)^(t) $= 2\pi k^2$ f(-t)

which then gives

Fact 3: f^^(t) $= 2\pi k^2$ f(-t) (C.5)

FT of f(t) = f^(t) ⇔ FT of f^(t) $= 2\pi k^2$f(-t)

The first line says that applying the Fourier transform twice to a function gives $2\pi k^2$ times the function of negated argument (see Stakgold Vol II (5.55) with k = 1). Nothing new is happening here, it is all just notation. When $k = 1/\sqrt{2\pi}$ the factor $2\pi k^2 = 1$ on the second line which is a strong motivation for that scaling, but we had other motivations for k = 1 as outlined in Section 5.

The three Facts just stated are independent of the sign of the phase in the Fourier transform definition.

Operator Notation. An alternative notation similar to that used for Laplace transforms is the following:

f^(ω) = $\mathcal{F}$[f(u),ω] = $\mathcal{F}$[f,ω] = $\mathcal{F}$ f(ω)

or

f^ = $\mathcal{F}$ f (C.6)

and for the inverse Fourier transform,

$f^{\wedge -1}(t) = \mathcal{F}^{-1}[f(u),t] = \mathcal{F}^{-1}[f,t] = \mathcal{F}^{-1}f(t)$

or

$f^{\wedge -1} = \mathcal{F}^{-1}f$. (C.7)

The idea here is that $\mathcal{F}$ f = g is a new function obtained by acting upon function f with the Fourier transform operator $\mathcal{F}$. Similarly $\mathcal{F}^{-1}$f = h is a new function obtained by acting upon function f with the inverse Fourier transform operator $\mathcal{F}^{-1}$. The three facts stated above can then be translated into this new notation.

Fact 1: $f^{\wedge -1}(u) = (1/2\pi k^2)$ f^(-u) → $\mathcal{F}^{-1}[f(s),u] = (1/2\pi k^2)\,\mathcal{F}[f(s),-u]$
$\mathcal{F}^{-1}[f,u] = (1/2\pi k^2)\,\mathcal{F}[f,-u]$
$\mathcal{F}^{-1}f(u) = (1/2\pi k^2)\,\mathcal{F}\,f(-u)$ *(C.3)*

Fact 2: $[f^\wedge(\omega)]^{\wedge -1}(t) = [f^{\wedge -1}(\omega)]^\wedge(t) = f(t)$ → $\mathcal{F}^{-1}[\mathcal{F}[f(\omega),s],t] = \mathcal{F}[\mathcal{F}^{-1}[f(\omega),s],t] = f(t)$
$[f^\wedge]^{\wedge -1} = [f^{\wedge -1}]^\wedge = f$ → $\mathcal{F}^{-1}\mathcal{F}\,f = \mathcal{F}\mathcal{F}^{-1}f = f$ *(C.4)*

Fact 3: f^^(t) = 2πk f(-t) → $\mathcal{F}[\mathcal{F}[f(\omega),s],t] = 2\pi k^2 f(-t)$
$\mathcal{F}[\mathcal{F}[f,s],t] = 2\pi k^2 f(-t)$
$\mathcal{F}^2 f(t) = 2\pi k^2 f(-t)$ *(C.5)*

The last line of Fact 2 has a particular appeal, since $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathbf{1}$, the identity operator.

(b) Principal Value Integrals and the Tick Notation

Consider the following integral,

$$\int_{-\infty}^{\infty} dx \ \frac{1}{x-a} \ f(x) \ .$$

If f(x) is non-vanishing at x = a, and if a is real, this integral runs right through a pole at x = a. If a were *complex*, then the integral would run above or below the pole and one would be less concerned. If the intention really is to run the integration right through the pole, it is useful to make that fact very clear using some kind of notation. One defines the notion of "going through the pole" as the following limiting operation,

$$\int_{-\infty}^{\infty} dx \ \frac{1}{x-a} \ f(x) \ = \ \lim_{\varepsilon\to 0} \left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx \right] \ \frac{1}{x-a} \ f(x) \ \equiv \ ⨍_{-\infty}^{\infty} dx \ \frac{1}{x-a} \ f(x) \ . \qquad \text{(C.8)}$$

Such an integration is referred to as a **Cauchy Principal Value** (or Principal Part) Integral. We defer to section (d) below the pf notation which formalizes the above definition as

$$\int_{-\infty}^{\infty} dx \ \mathrm{pf}\left(\frac{1}{x-a}\right) f(x) \ \equiv \ ⨍_{-\infty}^{\infty} dx \ \frac{1}{x-a} \ f(x) \ \equiv \ \lim_{\varepsilon\to 0} \left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx \right] \ \frac{1}{x-a} \ f(x)$$

As an example, consider this case where a = 0 and f(x) = 1,

$$\int_{-2}^{2} dx \ \frac{1}{x} \ = \ 0 \ .$$

Although 1/x "blows up" at x = 0, this integral as defined above is exactly 0. All contributions to this integral are real, since 1/x is real, so the integral has no imaginary part. A simple argument for result zero is that the integration range is even while the integrand is odd under x → -x, so contributions from the left side of x=0 exactly cancel those from the right side of x=0. If the integrand contained some f(x) which was even under x → -x, the same argument would apply and the integral would be 0, but for general f(x) the integral would not be 0.

<u>Comment</u>: If one tries to evaluate the integral using ln(x), one gets a pre-limit result $\ln[\frac{a-\varepsilon}{a+\varepsilon}] + \ln[\frac{2}{-2}]$. After the limit, the first term gives ln[1] = 0 while the second term seems to give ln[-1] which one thinks of as ±iπ and we get a contradictory result that the real integral of a real integrand has an imaginary part. The problem is that this is a singular integral and the normal rules do not apply. The ±iπ reflects the fact that the integral is trying to avoid the pole by going above it or below it, whereas we really want to go right through it. Stakgold Vol. I Exercise 1.23 shows how this contradiction is resolved using a redefined log function, and the subject reappears below in our Pole Avoidance Rule discussion.

Here then are two notations used for integrals intended to run through poles,

$$⨍_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x)\ \equiv\ \text{P.V.} \int_{-\infty}^{\infty} dx\ \frac{1}{x-a}\ f(x)\ \equiv\ \lim_{\varepsilon\to 0} \left[\int_{-\infty}^{a-\varepsilon} dx + \int_{a+\varepsilon}^{\infty} dx \right] \frac{1}{x-a}\ f(x)\ . \qquad \text{(C.9)}$$

The tick mark on the integration symbol suggests the idea of running through the pole as the ε limit from the two sides, but is not easy to typeset, so one often sees the letters P.V, PV, p.v., v.p. , P or some other set of letters to indicate the principle part integration. We shall use the tick mark notation. Our example is then

$$⨍_{-2}^{2} dx\ \frac{1}{x}\ =\ 0\ . \qquad \text{(C.10)}$$

In the next several sections we shall examine some closely related examples of Fourier transforms, some of which require use of the Principal Value integral. The examples are later summarized in section (g).

(c) Example: f(u) = 1/u

<u>Projection/Transform</u>:

Let f(u) = 1/u. Using the regular Fourier transform requires that the integral go right through the pole, so we have

$$(1/u)\hat{}(\omega)\ =\ k \int_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u}\ = k\ ⨍_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u}\ . \qquad \text{(C.11)}$$

For $\omega \neq 0$ we can evaluate the integral this way,

$$⨍_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u}\ =\ ⨍_{-\infty}^{\infty} du(1/u)\ [-i\sin(\omega u)] = (-i)\ 2 \int_{0}^{\infty} du\ \sin(\omega u)/u$$

$$= (-i)\ 2\ \text{sign}(\omega)\ \{ \int_{0}^{\infty} du\ \sin(|\omega|u)/u\ \}\ \ = (-i)\ 2\ \text{sign}(\omega)\ \{ \int_{0}^{\infty} dx\ \text{sinc}(x)\ \} \qquad // \ x = |\omega|u$$

$$= (-i)\ 2\ \text{sign}(\omega)\ \ \{ \pi/2 \}$$

$$= -i\pi\ \text{sign}(\omega)\ . \qquad \text{(C.12)}$$

In the first step cos(ωu) was discarded since (1/u) cos(ωu) is an odd function of u. Once that is done, since sinc(x) = 1 at x = 0, there is no longer a pole at u = 0 so we have just a regular integral. The residual integral is half of (10.3).

If ω = 0, the integral is just $⨍_{-\infty}^{\infty} du(1/u) = 0$ based on the discussion of the previous section. One can combine these results by writing

$$⨍_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u}\ =\ -i\pi\ \text{sgn}(\omega) \qquad \text{(C.13)}$$

where

$$\text{sgn}(\omega) \equiv \begin{cases} +1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases} \qquad \text{// sometimes called signum}(\omega) \qquad \text{(C.14)}$$

This sgn(ω) function is related to the Heaviside step function by

$$\text{sgn}(\omega) = 2\theta(\omega) - 1 \qquad \text{(C.15)}$$

where in particular sgn(0) = 2θ(0) – 1 = 2(1/2)-1 = 0.

Our conclusion is that the Fourier transform of 1/u is given by

$$(1/u)\hat{}(\omega) = -i\pi k \,\text{sgn}(\omega) \ . \qquad \text{(C.16)}$$

If we treat sgn(ω) like any other function, we can suppress the ω argument to write

$$(1/u)\hat{} = -i\pi k \,\text{sgn} \ . \qquad \text{(C.17)}$$

In the general case of f^(u) → f^ we could suppress the u, but once the function is stated (such as 1/u), it is difficult to suppress the u and still know what the function is. Note that the u in 1/u is just a dummy variable and we could just as well write

$$(1/t)\hat{}(\omega) = -i\pi k \,\text{sgn}(\omega)$$
$$(1/t)\hat{} = -i\pi k \,\text{sgn} \ . \qquad \text{(C.18)}$$

<u>Inversion/Recovery</u>:

How does the recovery work?

$$(1/2\pi k) \int_{-\infty}^{\infty} d\omega \,(1/t)\hat{}(\omega)\, e^{+i\omega t} = (1/2\pi k) \int_{-\infty}^{\infty} d\omega \,[-i\pi k \text{sgn}(\omega)]\, e^{+i\omega t}$$

$$= (1/2\pi)(-i\pi) \int_{-\infty}^{\infty} d\omega \,\text{sgn}(\omega)\, e^{+i\omega t} = (-i/2) \int_{-\infty}^{\infty} d\omega \,\text{sgn}(\omega)\, [i \sin(\omega t)]$$

$$= (-i/2)\, i \ 2 \int_0^{\infty} d\omega \sin(\omega t) = \int_0^{\infty} d\omega \sin(\omega t) = -(1/t)\cos(\omega t)\Big|_0^{\infty} = -(1/t)\,[\,\cos(\infty t) - \cos(0)]$$

$$= (1/t) \qquad \text{(C.19)}$$

The distributional trick is to set cos(∞t) = 0. In more detail,

$$\int_0^{\infty} d\omega \sin(\omega t) = \lim_{\varepsilon\to 0} \left[\int_0^{\infty} d\omega \sin(\omega t)\, e^{-\varepsilon\omega} \right] = \lim_{\varepsilon\to 0} \frac{t}{t^2+\varepsilon^2} = \frac{1}{t} \ . \qquad \text{(C.20)}$$

Interchange:

We have just shown that

$(1/u)\hat{}(\omega) = -i\pi k \,\mathrm{sgn}(\omega)$.

We can Fourier transform both sides to get

$[(1/u)\hat{}(\omega)]\hat{}(t) = -i\pi k \,[\mathrm{sgn}(\omega)]\hat{}(t)$

so that

$[\mathrm{sgn}(\omega)]\hat{}(t) = (1/-i\pi k)\ [(1/u)\hat{}(\omega)]\hat{}(t)$.

But according to Fact 3,

$[(1/u)\hat{}(\omega)]\hat{}(t) = [(1/u)\hat{}]\hat{}(t) = (1/u)\hat{}\hat{}(t) = 2\pi k^2\ (1/-t) = -2\pi k^2/t$.

Therefore

$$[\mathrm{sgn}(\omega)]\hat{}(t) = (1/-i\pi k)\ (-2\pi k^2/t) = 2k/(it) = -2ik(1/t) . \qquad \text{(C.21)}$$

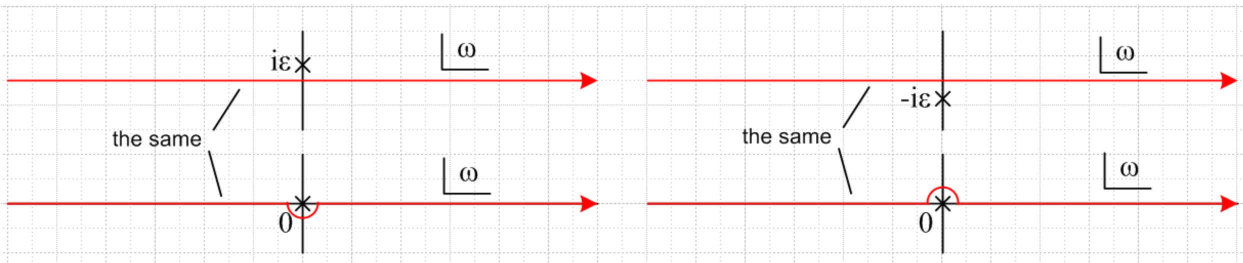
Thus we have learned the Fourier transform of the function sgn(ω). We could have computed this directly as follows:

$$[\mathrm{sgn}(\omega)]\hat{}(t) = k \int_{-\infty}^{\infty} du\ \mathrm{sgn}(u)\ e^{-itu} = k \int_{-\infty}^{\infty} du\ \mathrm{sgn}(u)\ [-i \sin(tu)] = (-i)\ 2k \int_{0}^{\infty} du\ \sin(tu)$$

$$= (-2ik)(-1/t)\cos(tu)|^{\infty}_{0} = (2ik/t)(0 - 1) = -2ik(1/t) .$$

(d) The Pole Avoidance Rule of Complex Integration

The upper picture on the left shows a real-axis integration contour in the ω-plane which passes just *below* a pole located at ω = +iε. We are interested in what happens as ε → 0. The upper picture on the right shows the contour passing just *above* a pole at ω = -iε and we have a similar interest there as ε → 0. The lower contour pictures show a certain contour deformation, and the pair of equations under each pair of drawings will be discussed below.



$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, \frac{1}{\omega - i\varepsilon} = \fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) + i\pi\, f(0) \qquad \int_{-\infty}^{\infty} d\omega\, f(\omega)\, \frac{1}{\omega + i\varepsilon} = \fint_{-\infty}^{\infty} d\omega\, f(\omega)(1/\omega) - i\pi\, f(0)$$

$$\frac{1}{\omega - i\varepsilon} = \mathrm{pf}(1/\omega) + i\pi\delta(\omega) \qquad \frac{1}{\omega + i\varepsilon} = \mathrm{pf}(1/\omega) - i\pi\delta(\omega)$$

Fig C.1

We assume that $f(\omega)$ is such that the integrals converge and $f(\omega)$ is well defined at $\omega = 0$. We are interested only in the limit $\varepsilon \to 0$.

<u>Left Side.</u> Consider the top left red arrow contour. If we try to take $\varepsilon \to 0$, the pole moves down and hits the contour, which is a poorly defined concept in complex integration. To prevent this from happening, we first make a tiny semi-circular deformation of the contour so it goes around $\omega = 0$. One is certainly allowed to deform a contour and not change an integral, as long as the deformation hits no singularities. After doing this "for free" deformation, we *then* let the $\varepsilon \to 0$ so the pole moves down to the real axis. If we now evaluate the integral, we get two famous pieces: The first piece is the principle value integral discussed in section (b) above, namely,

$$\fint_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega) \equiv \lim_{\alpha \to 0} \left[\int_{-\infty}^{-\alpha} + \int_{\alpha}^{\infty} \right] d\omega\, f(\omega)\, (1/\omega)\,.$$

The second piece is a half-circle counterclockwise contour around the pole which gives *one half* the pole residue which result is then $(1/2)\, 2\pi i\, f(0) = i\pi f(0)$. In general, if a contour goes some percentage around a pole, it picks up that percentage of the reside, which we now demonstrate, letting $\omega = Re^{i\theta}$,

$$\oint \frac{d\omega}{\omega} = \oint \frac{Re^{i\theta} i d\theta}{Re^{i\theta}} = i \int_{\theta_1}^{\theta_2} d\theta = i(\theta_2 - \theta_1)\,. \qquad \text{// partial residue rule}$$

If we go half way around the pole, then $i(\theta_2 - \theta_1) = i\pi$. So we have now derived the top left equation in Fig C.1.

Recall now from Appendix A that a distributional equation is one which gains its meaning when placed inside an integral. So consider

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, \frac{1}{\omega - i\varepsilon} = \int_{-\infty}^{\infty} d\omega\, f(\omega)\, [\ \mathrm{pf}(1/\omega) + i\pi\delta(\omega)\ ] = \int_{-\infty}^{\infty} d\omega\, f(\omega)\, \mathrm{pf}(1/\omega) + i\pi\, f(0)\,.$$

Here both pf(1/ω) and δ(ω) are symbolic functions as discussed in Appendix A. The meaning of δ(ω) seems clear (sifting property) while the meaning of pf(1/ω) is precisely this:

$$\int_{-\infty}^{\infty} d\omega\, f(\omega)\, \mathrm{pf}(1/\omega) \;\equiv\; ⨍_{-\infty}^{\infty} d\omega\, f(\omega)\, (1/\omega)\ .$$

The letters pf stand for pseudofunction. Officially f(ω) should be a distribution theory "test function", but we just take it to be any reasonable function as described above. So we have now derived both equations on the left of Fig C.1.

Right Side. This is the same idea, but the required contour deformation is different, and since the semicircle then goes clockwise around the pole, we pick up minus half the residue which is – iπf(0). Now both equations on the right are derived. We have then proven the following distributional equation:

The Pole Avoidance Rule:

$$\lim_{\varepsilon\to 0} \frac{1}{\omega \mp i\varepsilon} = \mathrm{pf}(1/\omega) \pm i\pi\delta(\omega) \qquad \text{(C.22)}$$

Notice that this rule has no connection with phase sign conventions of the Fourier transform. It really has nothing at all to do with the Fourier transform in fact. This "rule" seems to have no official name, so we have made one up. For more discussion of this subject see Stakgold Vol. I page 50 (1.27) and previous pages.

(e) Example: f(u) = 1/(u±iε)

Here we always imply the limit ε→ 0.

Projection/Transform:

Using the Pole Avoidance Rule (C.21) above, we may compute the Fourier transform of this f(u) as follows:

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = k \int_{-\infty}^{\infty} du \frac{1}{u\pm i\varepsilon}\, e^{-i\omega u} = k ⨍_{-\infty}^{\infty} du (1/u)\, e^{-i\omega u} \mp i\pi k$$

The principal value integral was found in (C.13) to be -iπ sgn(ω) , so we find that

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = -i\pi k\, \mathrm{sgn}(\omega) \mp i\pi k = -i\pi k\, (\mathrm{sgn}(\omega) \pm 1)\ .$$

Assume first the upper signs,

$$(\frac{1}{u+ i\varepsilon})\hat{}(\omega) = -i\pi k\, \mathrm{sgn}(\omega) - i\pi k = -i\pi k\, (\mathrm{sgn}(\omega) + 1)$$

If $\omega > 0$, the result is $-2\pi ik$, and if $\omega < 0$ the result is 0. So

$$(\frac{1}{u+i\varepsilon})\hat{}(\omega) = -2\pi ik\theta(\omega) .$$

Now assume the lower signs

$$(\frac{1}{u-i\varepsilon})\hat{}(\omega) = -i\pi k \,\mathrm{sgn}(\omega) + i\pi k = -i\pi k (\mathrm{sgn}(\omega) - 1)$$

If $\omega > 0$, the result is 0. If $\omega < 0$, the result is $+2\pi ik$. So

$$(\frac{1}{u-i\varepsilon})\hat{}(\omega) = 2\pi ik\theta(-\omega) .$$

Combining these results we get the Fourier transform of $\frac{1}{u\pm i\varepsilon}$:

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = \mp 2\pi ik\, \theta(\pm\omega) \quad . \tag{C.23}$$

<u>Inversion/Recovery:</u>

$$(1/2\pi k) \int_{-\infty}^{\infty} d\omega\, (\frac{1}{u\pm i\varepsilon})\hat{}(\omega)\, e^{+i\omega t} = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, [\mp 2\pi i\, \theta(\pm\omega)]\, e^{+i\omega t}$$

$$= (1/2\pi)(\mp 2\pi i) \int_{-\infty}^{\infty} d\omega\, \theta(\pm\omega)\, e^{+i\omega t} = (\mp i) \int_{-\infty}^{\infty} d\omega\, \theta(\pm\omega)\, e^{+i\omega t}$$

First take the upper sign

$$= (-i) \int_{0}^{\infty} d\omega\, e^{+i\omega t} \quad .$$

If t has a small positive imaginary part, the integral converges to $(-1/it)$ to give

$$= (-i)(-1/it) = 1/t$$

but we write t as $t+i\varepsilon$ to show that it has this small positive imaginary part, so the result is

$$(1/2\pi) \int_{-\infty}^{\infty} d\omega\, (\frac{1}{u+i\varepsilon})\hat{}(\omega)\, e^{+i\omega t} = \frac{1}{t+i\varepsilon}$$

as desired. Now we look at the lower sign

$$(+i) \int_{-\infty}^{0} d\omega\, e^{+i\omega t} = (+i) \int_{0}^{\infty} d\omega\, e^{-i\omega t}$$

If t has a small negative imaginary part, the integral converges to (+1/it) to give

$$= (+i)\,(+1/it) = 1/t = \frac{1}{t-i\varepsilon}$$

which is again the desired result.

Interchange:

We have just shown that

$$(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = \mp\, 2\pi ik\ \theta(\pm\omega)$$

where it is understood that $\varepsilon \rightarrow 0$ on the left side, and either set of signs is valid.

We can Fourier transform both sides to get

$$[(\frac{1}{u\pm i\varepsilon})\hat{}(\omega)]\hat{}(t) = \mp\, 2\pi ik\ [\theta(\pm\omega)]\hat{}(t)\ .$$

But according to Fact 3,

$$[(\frac{1}{u\pm i\varepsilon})\hat{}(\omega)]\hat{}(t) = [(\frac{1}{u\pm i\varepsilon})\hat{}]\hat{}(t) = (\frac{1}{u\pm i\varepsilon})\hat{}\hat{}(t) = 2\pi k^2 \frac{1}{-t\pm i\varepsilon}\ .$$

Therefore

$$[\theta(\pm\omega)]\hat{}(t) = (\mp\, 2\pi ik)^{-1}\, 2\pi k^2 \frac{1}{-t\pm i\varepsilon} = \frac{k}{\mp i}\ \frac{1}{-t\pm i\varepsilon} = \frac{k}{\pm it+\varepsilon}$$

$$= \pm ik \frac{1}{-t\pm i\varepsilon} = \pm ik\ [\ pf(-1/t) \mp i\pi\delta(-t)\ ] = \pm ik\ [\ -pf(1/t) \mp i\pi\delta(t)\ ] = \mp\ i\ k\ pf(1/t) + \pi\ k\ \delta(t)$$

and so we learn the Fourier transform of the Heaviside step function with either sign argument. The result with the + sign is verified immediately below. A more standard naming of the arguments yields

$$[\theta(\pm t)]\hat{}(\omega) = \pm i\ k \frac{1}{-\omega\pm i\varepsilon} = \mp\ i\ k\ pf(1/\omega) + \pi\ k\ \delta(\omega) = k\ pf(\frac{1}{\pm i\omega}) + \pi\ k\ \delta(\omega)\ . \qquad (C.24)$$

(f) Example: f(u) = θ(u) using the Generalized Fourier Transform

We include this example only because it is related to the previous examples.

Projection/Transform:

From the first line of (C.1) the projection (Fourier transform) is given by

$$[\theta(u)]^\wedge(\omega) = k \int_{-\infty}^{\infty} du\, \theta(u)\, e^{-i\omega u} = k \int_{0}^{\infty} du\, e^{-i\omega u} \ .$$

The generalized Fourier transform was defined in Section 6. Recall that the recovery contour in the inversion formula passes below all singularities of f(u), so for this example that contour runs just below the real axis so as to put the pole at u = 0 above the contour. Thus, we are really interested in the projection evaluated at ω-iε, so we then have a convergent integral,

$$[\theta(u)]^\wedge(\omega-i\varepsilon) = k \int_{0}^{\infty} du\, e^{-i(\omega-i\varepsilon)u} = k \int_{0}^{\infty} du\, e^{-(i\omega+\varepsilon)u} = k\, \frac{1}{i\omega+\varepsilon}\, .$$

Taking the limit ε→ 0 we obtain the generalized Fourier transform of θ(u) valid for general complex ω,

$$[\theta(u)]^\wedge(\omega) = \frac{k}{i\omega} \qquad \text{(C.25)}$$

Inversion/Recovery:

We now evaluate the generalized Fourier inversion formula using the above projection,

$$(1/2\pi k) \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} d\omega\, f(\omega)\, e^{+i\omega t} = (1/2\pi k) \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} d\omega\, \frac{k}{i\omega}\, e^{+i\omega t}$$

$$= (1/2\pi i) \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} d\omega\, \frac{1}{\omega}\, e^{+i\omega t} = \theta(t) \qquad \text{as explained below :}$$

For t < 0, we close the contour down and pick up nothing giving 0.
For t > 0, we close the contour up and pick up the full pole residue to get (1/2πi)2πi= 1.

(g) Summary of Examples

Our convention uses k = 1, but see (C.1) for other conventions.

Function/Distribution	Fourier transform	Comment
x(t)	X(ω)	main text notation (k=1)
f(t)	$f^{h}(\omega)$	Appendix C notation
1/t	-iπk sgn(ω)	*(C.16)*
sgn(t)	$\frac{2k}{i\omega}$	*(C.21)*
$\lim_{\varepsilon\to 0} \frac{1}{t\pm i\varepsilon} = pf(1/t) \mp i\pi\delta(t)$	$\mp\ 2\pi ik\ \theta(\pm\omega)$	*(C.23)*
θ(±t)	$k \lim_{\varepsilon\to 0}\frac{1}{\pm i\omega+\varepsilon} = k\ pf(\frac{1}{\pm i\omega}) + \pi k\ \delta(\omega)$	*(C.24)*
θ(t)	$\frac{k}{i\omega}$ (generalized FT)	*(C.25)*

Comment: Looking back at (C.11) in light of the pf notation, the formally correct version of (C.11) would be this

$$[pf(1/u)]\hat{}(\omega) = k \int_{-\infty}^{\infty} du\ pf(1/u)\ e^{-i\omega u} = k \fint_{-\infty}^{\infty} du(1/u)\ e^{-i\omega u} \qquad (C.11)$$

and then one would have this version of (C.18).

$$[pf(1/t)]\hat{}(\omega) = -i\pi k\ sgn(\omega) \ . \qquad (C.18)$$

Most tables of Fourier transforms omit the pf formality since things are clear without such notation.

(h) The Hilbert Transform and its relation to the Fourier Transform

The Hilbert Transform is defined as follows

$$f^{h}(t) \equiv (1/\pi) \fint_{-\infty}^{\infty} d\omega \frac{f(\omega)}{t-\omega} \equiv \mathcal{H}[\ f(x),t] = \mathcal{H}[f,t] \ . \qquad (C.26)$$

Again we show several different notations, though we shall mainly use $f^{h}(t)$. Changing the integration variable from ω to t' gives

$$f^{h}(t) = \fint_{-\infty}^{\infty} dt' \frac{(1/\pi)}{t-t'}\ f(t') = \int_{-\infty}^{\infty} dt'\ pf(\frac{(1/\pi)}{t-t'})\ f(t') \qquad (C.27)$$

where we use the pf symbolic function introduced earlier. We recognize this as having the usual convolution form (3.1),

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \text{sometimes written} \qquad a = b * c \tag{3.1}$$

where a = b * c becomes,

$$f^h = pf(\frac{1}{\pi t}) * f \quad . \tag{C.28}$$

The Convolution Theorem was stated in (3.6)

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega)\, C(\omega) \tag{3.6}$$

or in our new notation, where for example a^(ω) = kA(ω),

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad a\hat{}(\omega) = k^{-1}\, b\hat{}(\omega)\, c\hat{}(\omega) \quad . \tag{3.6}$$

Therefore we obtain this diagonalized form of (C.27) in ω-space,

$$[f^h]\hat{}(\omega) = k^{-1}\, (\frac{1}{\pi t})\hat{}(\omega)\;\; f\hat{}(\omega) \tag{C.29}$$

But from (C.16) we know that (see Comment at the end of section g above)

$$(\frac{1}{\pi t})\hat{}(\omega) = -i\, k\, \text{sgn}(\omega) \tag{C.30}$$

Therefore (C.29) becomes (the scaling factor is now gone since both sides are Fourier transforms),

$$(f^h)\hat{}(\omega) = -i\, \text{sgn}(\omega)\, f\hat{}(\omega) \quad . \tag{C.31}$$

This well-known result relates the Fourier transform of the Hilbert transform of a function directly to the Fourier transform of that function. We now make immediate use of this equation.

Since (C.31) can be applied to any function f (always assuming the Hilbert integral converges), we apply it first to g,

$$[(g)^h]\hat{}(\omega) = -i\, \text{sgn}(\omega)\, (g)\hat{}(\omega) \tag{C.32}$$

and then we set $g = f^h$ to get

$$[(f^h)^h]\hat{}(\omega) = -i\, \text{sgn}(\omega)\, (f^h)\hat{}(\omega) \quad . \tag{C.33}$$

Installing (C.31) into the right side gives

$$[f^{hh}]\hat{}(\omega) = -i\, \text{sgn}(\omega)\, [\, -i\, \text{sgn}(\omega)\, f\hat{}(\omega)] = -\, f\hat{}(\omega). \tag{C.34}$$

Applying the inverse Fourier transform to both sides then gives

$$f^{hh}(t) = -f(t)$$

or

$$f^{hh} = -f \tag{C.35}$$

or in operator notation,

$$\mathcal{H}\,\mathcal{H}\,f = -f \ . \tag{C.36}$$

This says that application of the Hilbert transform twice to a function gives that function preceded by a minus sign (compare to Fact 3). We can apply $\mathcal{H}^{-1}$ to both sides to get

$$\mathcal{H}^{-1} f = -\,\mathcal{H} f \tag{C.37}$$

and, in so doing, we have discovered the formula for the *inverse* Hilbert transform,

$$\mathcal{H}^{-1}f(t) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{f(\omega)}{t-\omega} \equiv f^{h-1}(t) \ . \tag{C.38}$$

We can then apply this last equation to f^h instead of f to get

$$\mathcal{H}^{-1}f^h(t) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{f^h(\omega)}{t-\omega} = (f^h)^{h-1}(t) = f(t) \tag{C.39}$$

which gives us this Hilbert transform pair (in which the two members differ only by a sign),

$$f^h(t) = (1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{f(\omega)}{t-\omega} \qquad \text{// projection = transform}$$

$$f(t) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{f^h(\omega)}{t-\omega} \qquad \text{// inversion = recovery} \ . \tag{C.40}$$

We now make these replacements $\omega\rightarrow\omega'$, $f \rightarrow X$, $f^h \rightarrow X_h$, $t\rightarrow\omega$ to get

$$X_h(\omega) = (1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega-\omega'} \qquad \text{// projection = transform}$$

$$X(\omega) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X_h(\omega')}{\omega-\omega'} \qquad \text{// inversion = recovery} \tag{C.41}$$

and this is a more familiar statement of the Hilbert transform pair. We have proved that this transform is valid by making use of its connection to the Fourier transform shown in (C.31).

About 13 pages of Hilbert transforms appear in Erdélyi ET2.

Example 1: Compute the Hilbert Transform of $f(\omega) = e^{i\beta\omega}$:

$$f^h(t) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{1}{\omega-t} e^{i\beta\omega} = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{1}{\omega'} e^{i\beta(\omega'+t)}$$

$$= -(1/\pi)\ e^{i\beta t} ⨍_{-\infty}^{\infty} d\omega' \frac{1}{\omega'} e^{i\beta\omega'} = -(1/\pi)\ e^{i\beta t} ⨍_{-\infty}^{\infty} d\omega' \frac{1}{\omega'} (i) \sin(\beta\omega') \quad // \ x = \beta\omega'$$

$$= -(i/\pi)\ e^{i\beta t}\ 2\ \mathrm{sgn}(\beta) \int_0^{\infty} dx\ \mathrm{sinc}(x) = -(i/\pi)\ e^{i\beta t}\ 2\ \mathrm{sgn}(\beta)\ \{ \pi/2 \} \qquad // \text{ as in (C.12)}$$

$$= -i\ \mathrm{sgn}(\beta)\ e^{i\beta t}$$

Therefore

$$f(\omega) = e^{i\beta\omega} \quad \Leftrightarrow \quad f^h(t) = -i\ \mathrm{sgn}(\beta) e^{i\beta t} \qquad \text{(C.42)}$$

or in the notation of the main text

$$X(\omega) = e^{i\beta\omega} \quad \Leftrightarrow \quad X_h(\omega) = -i\ \mathrm{sgn}(\beta) e^{i\beta\omega} \qquad \text{(C.43)}$$

Example 2: Equation (C.42) is valid for any β, so it is valid for -β. Since the Hilbert transform is linear we can then superpose exponentials to get the following correspondences,

$$f(\omega) = k\ \Sigma_\beta\ g_\beta\ e^{-i\beta\omega} \quad \Leftrightarrow \quad f^h(t) = -ik\ \Sigma_\beta\ \mathrm{sgn}(-\beta)\ g_\beta\ e^{-i\beta t}$$

$$f(\omega) = k \int_{-\infty}^{\infty} d\beta\ g(\beta)\ e^{-i\beta\omega} \quad \Leftrightarrow \quad f^h(t) = -ik \int_{-\infty}^{\infty} d\beta\ \mathrm{sgn}(-\beta)\ g(\beta)\ e^{-i\beta t}$$

The last line can be written

$$f(\omega) = g\hat{}(\omega) \quad \Leftrightarrow \quad f^h(t) = +i\ [\ \mathrm{sgn}(\beta) g(\beta)\ ]\hat{}(t)$$

which then says

$$[g\hat{}(\omega)]^h(t) = +i\ [\ \mathrm{sgn}(\beta) g(\beta)\ ]\hat{}(t) \ .$$

If we suppress ω, then change t to ω, then β to t, and replace function name g by f, this says

$$(f\hat{})^h(\omega) = +i\ [\ \mathrm{sgn}(t) f(t)\ ]\hat{}(\omega) \qquad \text{(C.44)}$$

which we can compare with (C.31) which has the transforms in the reverse order

$$(f^h)\hat{}(\omega) = -i\ \mathrm{sgn}(\omega)\ f\hat{}(\omega) \ . \qquad \textit{(C.31)}$$

Alternate derivation of (C.44).

Start with (C.26) applied to f^ ,

$$(f\hat{})^{h}(t) \equiv (1/\pi) ⨍_{-\infty}^{\infty} d\omega \frac{f\hat{}(\omega)}{t-\omega}$$

or

$$(f\hat{})^{h} = pf(\frac{1}{\pi t}) * f\hat{}$$

The diagonalized version is then, using (C.30),

$$a\hat{}(\omega) = k^{-1} b\hat{}(\omega)\ c\hat{}(\omega)$$

or

$$((f\hat{})^{h})\hat{} = k^{-1} [-i\ k\ sgn(\omega)]\ f\hat{}\hat{}(\omega) = -i\ sgn(\omega)\ f\hat{}\hat{}(\omega) \ .$$

Use Fact 3 that $f\hat{}\hat{}(\omega) = 2\pi k^2 f(-\omega)$ to get

$$((f\hat{})^{h})\hat{}(\omega) = -i\ sgn(\omega)\ 2\pi k^2 f(-\omega) = 2\pi k^2\ i\ sgn(-\omega)\ f(-\omega)$$

Now take the Fourier transform of both sides

$$((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^2 i\ [sgn(-s)\ f(-s)]\hat{}(\omega)$$

Use Fact 3 that $((f\hat{})^{h})\hat{}\hat{}(\omega) = 2\pi k^2((f\hat{})^{h})(-\omega)$ to get

$$2\pi k^2((f\hat{})^{h})(-\omega) = 2\pi k^2 i\ [sgn(-s)\ f(-s)]\hat{}(\omega)$$

or

$$((f\hat{})^{h})(-\omega) = i[sgn(-s)\ f(-s)]\hat{}(\omega)$$

or

$$((f\hat{})^{h})(\omega) = i\ [sgn(-s)\ f(-s)]\hat{}(-\omega)$$

Finally we get to use Fact 0 that $[f(-u)]\hat{}(-\omega) = f\hat{}(\omega)$ to obtain the final result

$$((f\hat{})^{h})(\omega) = i\ [sgn(s)\ f(s)]\hat{}(\omega)$$

which is (C.44).

Appendix D: Calculation of a Sum which appears in (35.17)

We want to show that ($k = \omega T_1$, $z = e^{ik}$)

$$S \equiv \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1} z^{-s} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)} \tag{D.1}$$

where the right side is $2\pi\delta_6(k,N)$ of (A.20). There are no doubt elegant ways to verify this identity, but we shall use the tried and true brute force method. We start off:

$$S = \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1} z^{-s} = \sum_{s=-2N}^{2N} z^{-s} - \frac{1}{2N+1} \sum_{s=-2N}^{2N} |s|\, z^{-s} = S_1 - \frac{1}{2N+1} S_2 \ . \tag{D.2}$$

The sum S_1 we know from (A.30) and (A.15) is

$$S_1 = \sum_{s=-2N}^{2N} z^{-s} = 2\pi\, \delta_5(k,2N) = \frac{\sin[(N+1/2)k]}{\sin(k/2)} \quad . \tag{D.3}$$

So we work then on S_2 :

$$S_2 = \sum_{s=-2N}^{2N} |s|\, z^{-s} = \sum_{s=-2N}^{-1} |s|\, z^{-s} + \sum_{s=1}^{2N} |s|\, z^{-s} + 0$$

$$= \sum_{s=-2N}^{-1} (-s)\, z^{-s} + \sum_{s=1}^{2N} (s)\, z^{-s} = \sum_{s=1}^{2N} (s)\, z^{s} + \sum_{s=1}^{2N} (s)\, z^{-s} = \sum_{s=1}^{2N} s\,(z^{s} + z^{-s})$$

$$= 2 \sum_{s=1}^{2N} s \cos(ks) \ . \tag{D.4}$$

Suppose we define

$$S_3 \equiv \sum_{s=1}^{2N} \sin(ks) \quad . \tag{D.5}$$

Then (here ∂_k means d/dk)

$$S_2 = 2\, \partial_k S_3. \tag{D.6}$$

So we now work on S_3 :

$$S_3 = \sum_{s=1}^{2N} \sin(ks) = (1/2i) \sum_{s=1}^{2N} [\, z^s - z^{-s}] = (1/2i) [\sum_{s=1}^{2N} z^s - \text{c.c.}]$$

$$= (1/2i) [\, S_4 - \text{c.c.}] \qquad \text{(D.7)}$$

where c.c means complex conjugate. We next work on

$$S_4 = \sum_{s=1}^{2N} z^s = \sum_{s=0}^{2N} z^s - 1 . \qquad \text{(D.8)}$$

Change summation variable to r = s-N so this becomes, again using (A.30),

$$S_4 = \sum_{r=-N}^{N} z^{(r+N)} - 1 = z^N \sum_{r=-N}^{N} z^r - 1 = z^N\, 2\pi\, \delta_5(k,N) - 1 \qquad \text{(D.9)}$$

Then

$$[\, S_4 - \text{c.c.}] = \{ z^N\, 2\pi\, \delta_5(k,N) - 1\} - \{ z^{-N}\, 2\pi\, \delta_5(k,N) - 1\}$$
$$= (z^N - z^{-N})\, 2\pi\, \delta_5(k,N) . \qquad \text{(D.10)}$$

Backtracking now we find for (D.7) that

$$S_3 = (1/2i) [\, S_4 - \text{c.c.}] = (1/2i) (z^N - z^{-N})\, 2\pi\, \delta_5(k,N) = \sin(Nk)\, 2\pi\, \delta_5(k,N) . \qquad \text{(D.11)}$$

which gives a result we could have looked up (see Comment below),

$$\sum_{s=1}^{2N} \sin(ks) = \sin(Nk) \frac{\sin[(N+1/2)k]}{\sin(k/2)} . \qquad \text{(D.11)'}$$

It remains to compute S_2 according to (D.6),

$$S_2 = 2\, \partial_k\, S_3 = 2\, \partial_k[\sin(Nk)\, 2\pi\, \delta_5(k,N)] . \qquad \text{(D.12)}$$

Backing up more we then have from (D.2),

$$S = S_1 - \frac{1}{2N+1} S_2$$
$$= 2\pi\, \delta_5(k,2N) - \frac{1}{2N+1}\, 2\, \partial_k[\sin(Nk)\, 2\pi\, \delta_5(k,N)] \qquad \text{(D.13)}$$

so that

$$(2N+1)S = (2N+1)\, 2\pi\, \delta_5(k,2N) - 2\partial_k[\sin(Nk)\, 2\pi\, \delta_5(k,N)] . \qquad \text{(D.14)}$$

Anticipating denominators of powers up to $\sin^2(k/2)$ we rewrite this as

$$\sin^2(k/2)\,(2N+1)S = (2N+1)\sin^2(k/2)[\,2\pi\,\delta_5(k,2N)] - 2\ \sin^2(k/2)\,\partial_k[\sin(Nk)\,2\pi\,\delta_5(k,N)]$$
$$= \qquad T1 \qquad - \qquad T2$$

Then let

$$D5(k,N) \equiv [2\pi\,\delta_5(k,N)] = \frac{\sin[(N+1/2)k]}{\sin(k/2)} \tag{A.15}$$

to get

$$T1 = (2N+1)\sin^2(k/2)\,D5(k,2N)$$

$$T2 = 2\sin^2(k/2)\,\partial_k[\sin(Nk)\,D5(k,N)]\ . \tag{D.15}$$

The statement (D.1) which we are trying to prove is now this:

$$S = 2\pi\delta_6(n,K) = \frac{1}{2N+1}\,\frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)} \tag{D.1}$$

and adding our factors to both sides this becomes

$$\sin^2(k/2)\,(2N+1)S = \sin^2(k/2)\,(2N+1)\ 2\pi\delta_6(n,K)$$

$$= \sin^2(k/2)\,(2N+1)\left\{\frac{1}{2N+1}\,\frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)}\right\}$$

$$= \sin^2[(N+1/2)k]\ . \tag{D.16}$$

If we then define this last factor as

$$T3 \equiv \sin^2[(N+1/2)k] \tag{D.17}$$

<u>our task is then to show that</u>

$$T1 - T2 = T3$$

or

$$Q \equiv T1 - T2 - T3 = 0. \tag{D.18}$$

where

$$T1 = (2N+1)\sin^2(k/2)\,D5(k,2N)$$
$$T2 = 2\sin^2(k/2)\,\partial_k[\sin(Nk)\,D5(k,N)]$$
$$D5(k,N) \equiv \frac{\sin[(N+1/2)k]}{\sin(k/2)}$$
$$T3 \equiv \sin^2[(N+1/2)k]\ . \tag{D.19}$$

This is a task for Maple. We enter the quantities T1,T2,T3 and the function D5(k,N) :

```
T1 := (2*N+1)*sin(k/2)^2*D5(k,2*N);
```

$$T1 := (2\,N+1)\,\sin\left(\frac{1}{2}k\right)^2\,\mathrm{D5}(k,\,2\,N)$$

```
T2 := 2*sin(k/2)^2*Diff(sin(N*k)*D5(k,N),k);
```

$$T2 := 2\,\sin\left(\frac{1}{2}k\right)^2\left(\frac{\partial}{\partial k}\sin(N\,k)\,\mathrm{D5}(k,\,N)\right)$$

```
D5 := (k,N) -> sin((N+1/2)*k)/sin(k/2);
```

$$\mathrm{D5} := (k,\,N) \rightarrow \frac{\sin\left(\left(N+\frac{1}{2}\right)k\right)}{\sin\left(\frac{1}{2}k\right)}$$

```
T3 := sin((N+1/2)*k)^2;
```

$$T3 := \sin\left(\left(N+\frac{1}{2}\right)k\right)^2$$

```
Q := T1-T2-T3;
```

$$Q := (2\,N+1)\,\sin\left(\frac{1}{2}k\right)\sin\left(\left(2\,N+\frac{1}{2}\right)k\right) - 2\,\sin\left(\frac{1}{2}k\right)^2\left(\frac{\partial}{\partial k}\,\frac{\sin(N\,k)\,\sin\left(\left(N+\frac{1}{2}\right)k\right)}{\sin\left(\frac{1}{2}k\right)}\right) - \sin\left(\left(N+\frac{1}{2}\right)k\right)^2$$

The Maple value command causes the differentiation to be carried out, so we continue :

:

```
> Q1 := simplify(value(Q));
```

$$Q1 := 2\,\sin\left(\frac{1}{2}k\right)\sin\left(\frac{1}{2}(4\,N+1)\,k\right)N + \sin\left(\frac{1}{2}k\right)\sin\left(\frac{1}{2}(4\,N+1)\,k\right)$$
$$- 2\,\cos(N\,k)\,N\sin\left(\frac{1}{2}(2\,N+1)\,k\right)\sin\left(\frac{1}{2}k\right) - 2\,\sin(N\,k)\cos\left(\frac{1}{2}(2\,N+1)\,k\right)\sin\left(\frac{1}{2}k\right)N$$
$$- \sin(N\,k)\cos\left(\frac{1}{2}(2\,N+1)\,k\right)\sin\left(\frac{1}{2}k\right) + \sin(N\,k)\sin\left(\frac{1}{2}(2\,N+1)\,k\right)\cos\left(\frac{1}{2}k\right) - 1 + \cos\left(\frac{1}{2}(2\,N+1)\,k\right)^2$$

```
> Q2 := expand(Q1);
```

$$Q2 := 2\,\sin\left(\frac{1}{2}k\right)^2 N\cos(N\,k)^2 - 2\,\sin\left(\frac{1}{2}k\right)^2 N + 2\,\sin\left(\frac{1}{2}k\right)^2\cos(N\,k)^2 - \sin\left(\frac{1}{2}k\right)^2$$
$$+ 2\,\sin(N\,k)^2\sin\left(\frac{1}{2}k\right)^2 N + 2\,\sin(N\,k)^2\sin\left(\frac{1}{2}k\right)^2 + \sin(N\,k)^2\cos\left(\frac{1}{2}k\right)^2 - 1 + \cos(N\,k)^2\cos\left(\frac{1}{2}k\right)^2$$

```
> Q3 := simplify(Q2);
```

$$Q3 := 0$$

Thus we have shown that Q = 0 so (D.1) is then verified. **QED**

Comment: Gradshteyn and Ryzhik provide the following summation formulas :

1.342

1. $$\sum_{k=1}^{n} \sin kx = \sin \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$$

2.[10] $$\sum_{k=0}^{n} \cos kx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1$$
$$= \cos \frac{nx}{2} \sin \frac{n+1}{2} x \operatorname{cosec} \frac{x}{2} = \frac{1}{2}\left(1 + \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin \frac{x}{2}}\right)$$

1.352

1.[11] $$\sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos\left(\frac{2n-1}{2}x\right)}{2 \sin \frac{x}{2}}$$

2.[11] $$\sum_{k=1}^{n-1} k \cos kx = \frac{n \sin\left(\frac{2n-1}{2}x\right)}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}$$

An alternate method of verifying (D.1) would be to use two of these sums in an approach that begins this way

$$S \equiv \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1}\, z^{-s} = \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1}\, (z^{-s} + z^{s})/2 = \sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1} \cos(ks)$$

$$= 1 + 2 \sum_{s=1}^{2N} \frac{2N+1-s}{2N+1} \cos(ks) = 1 + 2 \sum_{s=1}^{2N} \cos(ks) - 2 \sum_{s=1}^{2N} s \cos(ks)$$

Appendix E: Table of Transforms

The following transforms appear in this document:

Fourier Integral Transform: x(t) aperiodic and continuous, X(ω) continuous, no image spectra

$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t}$ projection = transform *(1.1)*

$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t}$ expansion = inverse transform *(1.2)*

Generalized Fourier Integral Transform: recovery contour passes below all singularities of X(ω)

$X(\omega) = \int_{0}^{\infty} dt\, x(t)\, e^{-i\omega t}$ projection = transform *(6.4)*

$x(t) = (1/2\pi) \int_{-ci-\infty}^{-ci+\infty} d\omega\, X(\omega)\, e^{+i\omega t}$ expansion = inverse transform *(6.5)*

Fourier Cosine Transform:

$X_c(\omega) = 2 \int_{0}^{\infty} dt\, x(t) \cos(\omega t)$ projection = transform

$x(t) = (1/\pi) \int_{0}^{\infty} d\omega\, X_c(\omega) \cos(\omega t)$ expansion = inverse transform *(1.7)*

Fourier Sine Transform:

$X_s(\omega) = 2 \int_{0}^{\infty} dt\, x(t) \sin(\omega t)$ projection = transform

$x(t) = (1/\pi) \int_{0}^{\infty} d\omega\, X_s(\omega) \sin(\omega t)$ expansion = inverse transform *(1.8)*

Fourier Integral Transform in Appendix C notation: k is an arbitrary convention constant

$x\hat{}(\omega) = k \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t}$ projection = transform

$x(t) = (1/2\pi k) \int_{-\infty}^{\infty} d\omega\, x\hat{}(\omega)\, e^{+i\omega t}$ expansion = inverse transform

$f\hat{}(\omega) = k \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u}$ Fourier transform of f(u)

$f\hat{}^{-1}(t) = (1/2\pi k) \int_{-\infty}^{\infty} du\, f(u)\, e^{+iut}$ inverse Fourier transform of f(u) *(C.1)*

Fourier Series Transform: x(t) periodic and continuous with period T_1, spectrum is discrete

$$x(t) = \sum_{n=-\infty}^{\infty} x_{pulse}(t - nT_1) \qquad (14.1)$$

Complex form:

$$c_m \equiv (1/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t)\, e^{-im\omega_1 t} = (1/T_1) \int_{0}^{T_1} dt\, x(t)\, e^{-im\omega_1 t} \qquad (14.16)$$

$$x(t) = \sum_{m=-\infty}^{\infty} c_m\, e^{+im\omega_1 t} \qquad (15.1)$$

$$c_m = c(m\omega_1) = (1/T_1) X_{pulse}(m\omega_1) \qquad (14.8)\ \text{and}\ (14.10)$$

Real form:

$$a_m \equiv (2/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t) \cos(m\omega_1 t) = (2/T_1) \int_{0}^{T_1} dt\, x(t) \cos(m\omega_1 t) \qquad (15.6)$$

$$b_m \equiv (2/T_1) \int_{-\infty}^{\infty} dt\, x_{pulse}(t) \sin(m\omega_1 t) = (2/T_1) \int_{0}^{T_1} dt\, x(t) \sin(m\omega_1 t) \qquad (15.7)$$

$$x(t) = a_0/2 + \sum_{m=1}^{\infty} a_m \cos(m\omega_1 t) + \sum_{m=1}^{\infty} b_m \sin(m\omega_1 t) \qquad (15.9)$$

Laplace Transform: Right-sided (causal) x(t) vanishes for t < 0, x(t) and $\mathcal{X}(s)$ are continuous

$$\mathcal{X}(s) = \int_{0}^{\infty} dt\, x(t)\, e^{-st} \qquad \text{projection = transform} \qquad (6.9)$$

$$x(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{X}(s)\, e^{+is} \qquad \text{expansion = inverse transform} \qquad (6.10)$$

Relation to the Fourier Integral Transform:

$$\mathcal{X}(s) = X(s/i) \qquad X(\omega) = \mathcal{X}(i\omega) \qquad (6.8)$$

Digital Fourier Transform: $x(t_n)$ aperiodic, spectrum $X'(\omega)$ is continuous and contains image spectra

$$X'(\omega) \equiv \sum_{n=-\infty}^{\infty} \Delta t\, x(t_n)\, e^{-i\omega t_n} \qquad \text{projection = transform} \qquad (22.2)$$

$$x(t_n) = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega t_n} \qquad \text{expansion = inversion} \qquad (22.4)$$

or

$$X'(\omega) \equiv T_1 \sum_{n=-\infty}^{\infty} x_n\, e^{-i\omega n T_1} \qquad \text{projection = transform} \qquad x_n = x(t_n).$$

$$x_n = \frac{1}{2\pi} \int_{-\omega_1/2}^{\omega_1/2} d\omega\, X'(\omega)\, e^{+i\omega n T_1} \qquad \text{expansion = inversion}$$

where: $T_1 = \Delta t$ $\qquad \omega_1 = 2\pi/T_1 = 2\pi/\Delta t$ $\qquad t_n = n\,\Delta t = n\,T_1$.

Relation to the Fourier Integral Transform:

$$X'(\omega) = \sum_{m=-\infty}^{\infty} X(\omega - m\omega_1) = [\; X(\omega) + \sum_{m \neq 0} X(\omega - m\omega_1)\;] \qquad \text{// image spectra} \qquad (23.1)$$

Z Transform: $x(t_n) = x_n$ is aperiodic, spectrum $X''(z)$ is continuous and contains image spectra

$$X''(z) = \sum_{n=-\infty}^{\infty} x_n\, z^{-n} \qquad \text{projection = transform} \qquad (24.3)$$

$$x_n = \frac{1}{2\pi i} \int_C dz\, X''(z)\, z^{n-1} \qquad \text{expansion = inversion} \qquad (24.4)$$

where contour C goes once counterclockwise around the unit circle in the z-plane.

Relation to the Digital Fourier Transform and the Fourier Integral Transform:

$$X''(z) \equiv \frac{1}{T_1} X'(\omega) = \frac{1}{T_1} \sum_{m=-\infty}^{\infty} X(\omega - m\omega_1) \qquad z = e^{i\omega T_1} \qquad (24.1), (24.2)$$

Discrete Fourier Transform of a Pulse Train : $x(t_n)$ periodic, spectrum discrete, m integer

$$x(t_m) = \sum_{n=-\infty}^{\infty} x_{pulse}(t_m - nT_1)$$ sampled pulse train, $t_m = (m/N)T_1$

$$c'_m \equiv (1/N) \sum_{n=-\infty}^{\infty} x_{pulse}(t_n)\, e^{-imn(2\pi/N)}$$ projection = transform *(27.9)*

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)}$$ expansion = inverse transform *(27.11), (27.12)*

Discrete Fourier Transform (DFT) : A is an arbitrary convention constant

$$c'_m \equiv (A/N) \sum_{n=0}^{N-1} x_n\, e^{-imn(2\pi/N)}$$ $m = 0,1...N-1$ projection = transform

$$x_n = (1/A) \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)}$$ $n = 0,1,...N-1$ expansion = inverse transform *(27.22)*

Hilbert Transform:

$$X_h(\omega) = (1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X(\omega')}{\omega-\omega'}$$ projection = transform

$$X(\omega) = -(1/\pi) ⨍_{-\infty}^{\infty} d\omega' \frac{X_h(\omega')}{\omega-\omega'}$$ expansion = inverse transform *(C.41)*

Appendix F: The Spectrum and Power Density for Repeated-Sequence Pulse Trains

Appendix F contents: For infinite pulse trains composed of repeats of some length-P subsequence:

(a) computes X(ω) in (F.12)
(b) computes $\mathscr{P}(\omega)$ in (F.23) in terms of $|\, Y_P"(z)\,|^2$.
(c) computes $<\mathscr{P}(\omega)>$ for an ensemble of pulse trains which respect the special condition

$$<y_m^* y_n> = \alpha \quad \text{for } m \neq n$$
$$<y_m^* y_n> = \beta \quad \text{for } m = n \qquad s < |m-n| \qquad (F.25)$$

The result for $<\mathscr{P}(\omega)>$ is stated in (F.33) in several different forms.
(d) takes the P→∞ limit of the section (c) result for $<\mathscr{P}(\omega)>$
(e) computes $\mathscr{P}(\omega)$ for a single pulse train which respects the special condition

$$<y_m y_n>_1 = \alpha \quad \text{for } m \neq n + NP \qquad N = \text{any integer}$$
$$<y_m y_n>_1 = \beta \quad \text{for } m = n + NP \qquad (F.43)$$

where $<y_m y_n>_1$ is a horizontal average across the single sequence (autocorrelation).
The result for $\mathscr{P}(\omega)$ is stated in (F.52).
It is noted that the results for $<\mathscr{P}(\omega)>$ of (c) and $\mathscr{P}(\omega)$ of (e) are exactly the same in terms of their respectively defined α and β constants. It is then shown that these two sets of constants are the same.
(f) restates the overall result as the Fact (F.54).
A graphical representation is drawn for the spectrum in general, and then for a box pulse.
It is shown that the MLS sequence is a candidate for application of (F.54).
(g) treats the P = 2 repeated subsequence A,B using the general formulas of (a) and (b)

We start with this collection of equations:

$$Y"(z) = \sum_{n=-\infty}^{\infty} y_n e^{-i\omega n T_1} \qquad \text{Z Transform of } y_n \qquad (24.2) \qquad (F.1)$$

$$|\, Y"(z)\,|^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (F.2)$$

$$X(\omega) = X_{pulse}(\omega)\, Y"(z) \qquad (25.3) \qquad (F.3)$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega) \frac{T_1}{T} |\, Y"(z)\,|^2 \qquad (34.14) \qquad (F.4)$$

$$z = e^{i\omega T_1} \qquad \omega_1 \equiv 2\pi/T_1 \qquad (24.1)$$

In (F.4) T is the duration of the infinite pulse train, as in (33.22). The pulse train amplitudes are the y_n. Since sequence y_m is composed of subsequences of length P that repeat, we have this periodicity property of the y_n

$$y_{m+IP} = y_m \qquad \text{for any integer I} \qquad (F.5)$$

(a) Calculation of X(ω) for a Pulse Train with a Repeated Sequence

Consider first the sum in (F.1). Let n = IP + n' and write this sum as

$$\sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} = \sum_{I=-\infty}^{\infty} \sum_{n'=0}^{P-1} y_{IP+n'}\, e^{-i\omega(n'+IP)T_1} = \sum_{I=-\infty}^{\infty} e^{-i\omega IPT_1} \sum_{n'=0}^{P-1} y_{n'}\, e^{-i\omega n' T_1}$$

$$= \left(\sum_{I=-\infty}^{\infty} e^{-i\omega IPT_1}\right)\left(\sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1}\right) . \qquad \text{(F.6)}$$

We see that the sum factors into the product of two sums. The first sum we evaluate using

$$\sum_{n=-\infty}^{\infty} e^{-ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \quad . \qquad -\infty < k < \infty \qquad \textit{(A.31)}$$

Setting $k = \omega PT_1$ we find

$$\left(\sum_{I=-\infty}^{\infty} e^{-i\omega IPT_1}\right) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi m) \quad . \qquad \text{(F.7)}$$

The second sum we give the name $Y_P''(z)$ which is the Z transform of the subsequence $\{ y_0, y_1 ..... y_{P-1}\}$.

$$Y_P''(z) \equiv \sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1} \quad . \qquad \text{(F.8)}$$

Thus we have shown that

$$Y''(z) = \sum_{n=-\infty}^{\infty} y_n\, e^{-i\omega n T_1} = Y_P''(z) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi m) \quad , \qquad \text{(F.9)}$$

and then from (F.3).

$$X(\omega) = X_{pulse}(\omega)\, Y''(z) = X_{pulse}(\omega)\, Y_P''(z) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi m) \, . \qquad \text{(F.10)}$$

It is convenient to write

$$2\pi\delta(\omega PT_1 - 2\pi m) = 2\pi(PT_1)^{-1}\delta(\omega - 2\pi m/PT_1) = (1/P)\, \omega_1\, \delta(\omega - m\omega_1/P) \qquad \text{(F.11)}$$

and then

$$X(\omega) = X_{pulse}(\omega)\, \omega_1\, (1/P)\, Y_P''(z) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \tag{F.12}$$

$$\text{where} \quad Y_P''(z) \equiv \sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1} \quad . \tag{F.8}$$

$X(\omega)$ is the Fourier Transform Spectrum of an infinite pulse train composed of a repeating P-length subsequence. Since the pulse train is periodic, the spectrum is entirely discrete with lines at

$$\omega_m = (m/P)\omega_1. \tag{F.13}$$

(b) Calculation of $\mathscr{P}(\omega)$ for a Pulse Train with a Repeated Sequence

In Section 35 [see below (35.26)'] it was noted that $\mathscr{P}(\omega)$ can be calculated either by the Autocorrelation method or the Double Sum method. Here we shall take the latter method based on (F.2) above.

We can reorganize the double sum in (F.2) into a quadruple sum by defining :

$n = IP + n'$
$m = JP + m'$.

Then the double sum above becomes,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{JP+m'})^*\, (y_{IP+n'})\, e^{i\omega P(I-J)T_1}\, e^{i\omega(m'-n')T_1}$$

$$= \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} \sum_{n'=0}^{P-1} \sum_{m'=0}^{P-1} (y_{m'})^*\, (y_{n'})\, e^{i\omega P(I-J)T_1}\, e^{i\omega(m'-n')T_1} \quad , \tag{F.14}$$

where in the last line we have used the periodicity (F.5) of the y_n. The following illustration shows how, in this reorganization, we first sum over a square grid patch with n' and m', and then we sum over an array of those patches with I and J.

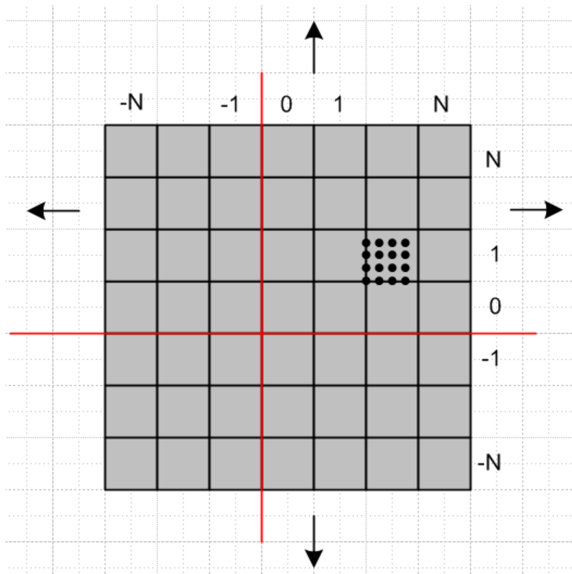


Fig F.1

In order to regulate things, we shall assume that the I and J sums range from -N to N rather than from -∞ to ∞. This means we are assuming that the sequence is (2N+1) repeated periods in length and not infinite. Then of course we can say

$$T = (2N+1)PT_1 . \qquad \text{(F.15)}$$

As usual, we keep N finite as long as possible, and then take N→ ∞ in the end.

We now rewrite (F.2) by removing the primes from summation indices and reordering the factors

$$| Y''(z) |^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \rightarrow$$

$$= [\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}] \, [\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^* \, y_n \, e^{i\omega(m-n)T_1}]$$

$$= [\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}] \, | Y_P''(z) |^2 \qquad \text{(F.16)}$$

As happened in (F.6) with a single sum, our double sum factors into a product of two double sums. The second double sum we recognize from (F.8) as $| Y_P''(z) |^2$. The first double sum is

$$[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1}] = | \sum_{I=-N}^{N} e^{i\omega PIT_1} |^2 \qquad \text{(F.17)}$$

Then apply (A.30)

$$\sum_{n=-N}^{N} e^{ink} = 2\pi\, \delta_5(k,N) = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \tag{A.30}$$

with $k = \omega PT_1$ to get

$$\left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] = \left| 2\pi\, \delta_5(\omega PT_1,N) \right|^2 = \left(2\pi\, \delta_5(\omega PT_1,N) \right)^2 . \tag{F.18}$$

Then from (A.20),

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} , \tag{A.20}$$

we can write the first factor of (F.16) as

$$\left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] = (2N+1)\, 2\pi\, \delta_6(\omega PT_1,N) = (1/P) \frac{T}{T_1} 2\pi\, \delta_6(\omega PT_1,N) \tag{F.19}$$

where T is from (F.15). At this point we have, looking at (F.16) and (F.19),

$$| Y''(z) |^2 = \left[\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \right] \left[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \right]$$

$$= \left[(1/P) \frac{T}{T_1} 2\pi\, \delta_6(\omega PT_1,N) \right] \; | Y_P''(z) |^2 . \tag{F.20}$$

Now finally we take $N\to\infty$ using (A.21)

$$\lim_{N\to\infty} \delta_6(k,N) = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \tag{A.21}$$

or

$$\lim_{N\to\infty} \delta_6(\omega PT_1,N) = \sum_{m=-\infty}^{\infty} \delta(\omega PT_1-2\pi m) \tag{F.21}$$

with this result

$$| Y''(z) |^2 = \left[(1/P) \frac{T}{T_1} 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega PT_1-2\pi m) \right] \; | Y_P''(z) |^2$$

or

$$\frac{T_1}{T} \, | Y''(z) | = (1/P) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega PT_1-2\pi m) \, | Y_P''(z) |^2$$

Using (F.11) we can rewrite this as

$$\frac{T_1}{T}\ |\ Y''(z)\ | = \omega_1\ (1/P)^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)\ \ |\ Y_P''(z)\ |^2\ . \tag{F.22}$$

Installing this into (F.4) then gives

$$\mathscr{P}(\omega)\ = \mathscr{P}_{pulse}(\omega)\ \omega_1\ (1/P)^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)\ \ |\ Y_P''(z)\ |^2 \tag{F.23}$$

$$\text{where}\quad Y_P''(z)\ \equiv \sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1}\quad . \tag*{\textit{(F.8)}}$$

$\mathscr{P}(\omega)$ is the Spectral Power Density of an infinite pulse train composed of a repeating P-length subsequence. The power density is discrete with lines at $\omega_m = (m/P)\omega_1$, the same lines observed in the spectrum $X(\omega)$ of (F.12).

(c) Calculation for an Ensemble of such Pulse Trains subject to Certain Conditions

We now imagine an ensemble of P-length subsequences s_i. We then create a corresponding ensemble of infinite length sequences S_i according to $S_i = \{.......s_i, s_i, s_i, s_i, ...\}$. This just an arbitrary ensemble, not a random ensemble or any other special kind ensemble.

Then we apply the ensemble average <..> to (F.23) to get (assuming now that the y_n are real),

$$<\mathscr{P}(\omega)>\ = \mathscr{P}_{pulse}(\omega)\ \ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P^2} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1}\ <y_m y_n>\ e^{i\omega(m-n)T_1} \tag{F.24}$$

At this point, *suppose it happens* that

$$<y_m y_n>\ = \alpha \qquad \text{for } m \neq n$$

$$<y_m y_n>\ = \beta \qquad \text{for } m = n \tag{F.25}$$

where α and β are *independent* of the indices shown. Notice in the (F.24) sum that max(n-m) = P-1, so we don't have to worry about these indices differing by an integral multiple of P. We have now restricted our interest to the sequence $\{y_0, y_1, ...y_{P-1}\}$.

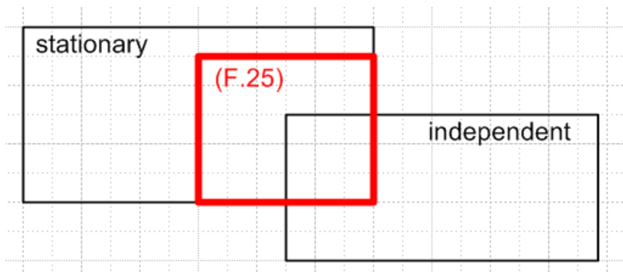
Comment: In Section 35 we showed in (35.11) that if a finite pulse train source has "stationarity", then

$$<y_m y_n>\ = f(n-m) \qquad \text{for } m \neq n$$
$$<y_m y_n>\ = f(0) \qquad \text{for } m = n \tag*{\textit{(35.11)}}$$

This is not sufficient to meet the condition (F.25). If in addition we assume that Y_n and Y_m are independent, we find from (35.15) that

$$f(n-m) = \begin{cases} \mu^2 & m \neq n \\ \sigma^2 + \mu^2 & m = n \end{cases} . \qquad \text{// stationarity and independence assumed} \qquad (35.15)$$

This *does* meet condition (F.25), but this assumption is more than we want to assume, so we just leave condition (F.25) as stated. Notice that if (F.25) is met, then (35.11) is valid so we are in the stationary realm. The general picture might be illustrated by this Venn diagram,



So assuming (F.25) we can write the double sum in (F.24) as, using $z = e^{i\omega T_1}$,

$$\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m y_n \rangle z^{m-n} = \alpha \sum_{n=0}^{P-1} \sum_{m\neq n} z^{m-n} + \beta \sum_{n=0}^{P-1} 1$$

$$= \alpha \sum_{n=0}^{P-1} \sum_{m\neq n} z^{m-n} + \beta P \qquad (F.26)$$

To evaluate the double sum, write it as

$$\sum_{n=0}^{P-1} \sum_{m\neq n} z^{m-n} = \sum_{n=0}^{P-1} [\sum_{m=0}^{P-1} z^{m-n} - \sum_{m=n} z^{m-n}] = \sum_{n=0}^{P-1} [\sum_{m=0}^{P-1} z^{m-n} - 1]$$

$$= \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} z^{m-n} - P = | \sum_{n=0}^{P-1} z^m |^2 - P = | \frac{1-z^P}{1-z} |^2 - P . \qquad (F.27)$$

Letting $z = e^{ik}$ with $k = \omega T_1$ one finds,

$$| \frac{1-z^P}{1-z} |^2 = \frac{(1-e^{-iPk})}{(1-e^{-ik})} \frac{(1-e^{iPk})}{(1-e^{ik})} = \frac{2-2\cos(kP)}{2-2\cos(k)} = \frac{1-\cos(kP)}{1-\cos(k)} \qquad (F.28)$$

so that

$$| \frac{1-z^P}{1-z} |^2 = \frac{\sin^2(Pk/2)}{\sin^2(k/2)} \quad . \tag{F.29}$$

Now recall from (A.20) that

$$\delta_6(k,N) \equiv \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \quad . \tag{A.20}$$

Setting N = (P-1)/2 we find P = 2N+1 and N+1/2 = P/2 so,

$$2\pi\delta_6(k, \frac{P-1}{2}) = \frac{1}{P} \frac{\sin^2(kP/2)}{\sin^2(k/2)} \quad . \tag{A.20}$$

and thus

$$| \frac{1-z^P}{1-z} |^2 = \frac{\sin^2(kP/2)}{\sin^2(k/2)} = P\, 2\pi\delta_6(k, \frac{P-1}{2}) \quad . \tag{F.30}$$

Therefore from (F.27),

$$\sum_{n=0}^{P-1} \sum_{m\neq n} z^{m-n} = 2\pi P\, \delta_6(k, \frac{P-1}{2}) - P = P\, [2\pi\delta_6(k, \frac{P-1}{2}) - 1\,] \tag{F.31}$$

and from (F.26)

$$\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} \langle y_m y_n \rangle\, z^{m-n} = \alpha P\, [2\pi\delta_6(k, \frac{P-1}{2}) - 1\,] + \beta P$$

$$= P\, [\, (\beta-\alpha) + \alpha\, 2\pi\delta_6(k, \frac{P-1}{2})\,] \tag{F.32}$$

the power density (F.24) becomes

$$\langle \mathscr{P}(\omega) \rangle = \mathscr{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P}\ [\, (\beta-\alpha) + \alpha\, 2\pi\delta_6(\omega T_1, \frac{P-1}{2})\,] \tag{F.33a}$$

where

$$2\pi\, \delta_6(\omega T_1, \frac{P-1}{2}) = \frac{1}{P} \frac{\sin^2(P\omega T_1/2)}{\sin^2(\omega T_1/2)} \, .$$

Evaluating at the delta function hit values $\omega = \omega_1 m/P$ ond finds,

$$2\pi\, \delta_6 = \frac{1}{P} \frac{\sin^2(m\pi)}{\sin^2(m\pi/P)} = \begin{cases} 0 & m \neq NP \\ P & m = NP \end{cases} \quad \text{for N = any integer} \ .$$

To get alternate forms for (F.33a), we first express each term as a separate sum,

$$<\mathcal{P}(\omega)> \ = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \ \omega_1\ \frac{1}{P}\ [(\beta\text{-}\alpha)\ \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha\ \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ 2\pi\delta_6(\omega T_1, \frac{P-1}{2})\ ]\ .$$

Since only those m which are multiples of P contribute to the second sum in (F.33a), we may rewrite that second sum as follows,

$$<\mathcal{P}(\omega)> \ = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \ \omega_1\ \frac{1}{P}\ [(\beta\text{-}\alpha)\ \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha\ \sum_{N=-\infty}^{\infty} \delta(\omega - \omega_1 N)\ P\ ] \qquad // \ m = NP$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\ \ \omega_1\ \ [(\beta\text{-}\alpha)\ \frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ \alpha\ \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.33b)}$$

$$= \ \mathcal{P}_{\mathbf{pulse}}(\omega)\ \ \omega_1\ \sum_{m=-\infty}^{\infty} [\ (\beta\text{-}\alpha)\ \frac{1}{P}\delta(\omega - \omega_1 m/P)\ +\ \alpha\ \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.33c)}$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1\ [\ (\beta\text{-}\alpha)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P)\ +\ \alpha\ \sum_{m \neq 0} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta\text{-}\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]\ . \qquad \text{(F.33d)}$$

If $x_{\mathbf{pulse}}(t)$ is real, then by (7.5) $\mathcal{P}_{\mathbf{pulse}}(\omega)$ is an even function of ω, and we can then reflect the negative part of the sum to the positive side to get,

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\ \omega_1\ [\ (\beta\text{-}\alpha)\frac{2}{P} \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m/P)\ +\ 2\alpha\ \sum_{m=1}^{\infty} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta\text{-}\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]\ . \qquad \text{(F.33e)}$$

In all forms of (F.33) we have: $\alpha = <y_m y_n>$ for $m \neq n$ $\quad \beta \ = <y_n^2>$.

These slightly different forms of $<\mathcal{P}(\omega)>$ are useful for different purposes. All results are valid for any finite integer P. Since each sequence in the ensemble is periodic with the same period P, the ensemble average spectrum is entirely discrete. The m≠0 sums include positive and negative integers.

(d) Limit as P → ∞ of the Ensemble Result

We would now like to take the limit of the above as P→∞. Write (F.33b) as

$$<\mathcal{P}(\omega)> = \mathcal{P}_{\mathbf{pulse}}(\omega)\ \ [(\beta\text{-}\alpha)\ \ \{\ \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \}\ +\ \omega_1\ \ \alpha\ \ \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)\ \ ] \quad . \qquad \text{(F.34)}$$

Then define

$$f_P(\omega) \equiv \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ . \qquad \text{(F.35)}$$

As $P\to\infty$, the spacing of the δ lines becomes closer and closer, while the amplitude $\frac{\omega_1}{P}$ of each δ line becomes less and less. Perhaps we can argue that in the limit this becomes some continuous function.

In line with the distribution theory approach to symbolic functions noted in Appendix A, suppose we integrate this function from some a to a+ε for small ε, where a is an arbitrary real number,

$$\int_a^{a+\varepsilon} f_P(\omega)\, d\omega = \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \int_a^{a+\varepsilon} \delta(\omega - \omega_1 m/P)$$

$$= \frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \Theta(a \le \omega_1 m/P \le a+\varepsilon) \qquad \text{(F.36)}$$

where we use the notation of Appendix A (e) for the Θ function which takes value 1 if the inequality argument is valid, meaning there is a delta hit. If P is a large integer, how many non-zero terms does this Σ_m have? The inequality argument reads

$$Pa/\omega_1 \le m \le Pa/\omega_1 + P\varepsilon/\omega_1 \ .$$

Since P is large, we round each term in this equation to the nearest integer, making little error. We select a very small ε first, and then we make sure P is large enough so $P\varepsilon/\omega_1$ is still a reasonably large integer when rounded. Then we have

$$\int_a^{a+\varepsilon} f_P(\omega)\, d\omega = \frac{\omega_1}{P} \sum_{m=Pa/\omega_1}^{Pa/\omega_1 + P\varepsilon/\omega_1} \Theta(a \le \omega_1 m/P \le a+\varepsilon) = \frac{\omega_1}{P} \sum_{m=Pa/\omega_1}^{Pa/\omega_1 + P\varepsilon/\omega_1} 1$$

$$= \frac{\omega_1}{P} \left(P\varepsilon/\omega_1 \right) = \varepsilon \ . \qquad \text{(F.37)}$$

Since we then have (for very large P) that $\int_a^{a+\varepsilon} f_P(\omega)\, d\omega = \varepsilon$ for any real a and for ε as small as we like, and since the integral over range ε is proportional to ε, the function $f_P(\omega)$ is equivalent to the constant function 1. Thus we have shown that,

$$\lim_{P\to\infty} f_P(\omega) = \lim_{P\to\infty} \left[\frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \right] = 1. \qquad \text{(F.38)}$$

We then obtain this $P\to\infty$ limit of (F.34),

$$<\mathscr{P}(\omega)> = \mathscr{P}_{\mathtt{pulse}}(\omega)\ \left[(\beta-\alpha) + \omega_1\ \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m) \right] \qquad \text{(F.39)}$$

$$= \mathscr{P}_{\mathtt{pulse}}(\omega)\ \left[(\beta-\alpha) + (1/T_1)\ \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_1 m) \right] \qquad // \ \omega_1 = 2\pi/T_1$$

$$= \mathcal{P}_{\mathbf{pulse}}(\omega)\ [(\beta\text{-}\alpha)\ +\ \alpha \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ ]\ .$$

This limit agrees with our result (35.11) for a random ensemble of infinite sequences for which $<a_m a_n>$ does not depend on the values of m and n,

$$<\mathcal{P}(\omega)> \ = \ \mathcal{P}_{\mathbf{pulse}}(\omega)\ \{\ (\beta\text{-}\alpha)\ +\alpha \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi m)\ \} \qquad \textit{(35.11)}$$

In the limit $P\to\infty$, the discrete lines $\delta(\omega - \omega_1 m/P)$ shown in (F.34) have coalesced into a continuous function. See Fig F.2 below for a graphical view.

(e) Calculation of $\mathcal{P}(\omega)$ for a single P-periodic Pulse Train subject to Certain Conditions

We really know ahead of time how this calculation will come out, but we do it nevertheless to convince the reader that the result is valid. After getting the result, we shall comment on why it is the way it is.

Recall this expression for $\mathcal{P}(\omega)$, which incorporates the discrete Wiener-Khintchine relation,

$$\mathcal{P}(\omega)\ = \mathcal{P}_{\mathbf{pulse}}(\omega)\,\frac{T_1}{T}\ |\ Y''(z)\,|^2\ = \mathcal{P}_{\mathbf{pulse}}(\omega)\ R''(z) \qquad \textit{(34.14a)} \qquad (F.40)$$

where R"(z) is the Z transform of the autocorrelation sequence r_s obtained from the y_n.

Our first task is to find r_s, which we defined this way,

$$r_s\ \equiv\ \lim_{N\to\infty} [\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s}\ ]\ =\ <y_n\, y_{n+s}>_1\ . \qquad \textit{(32.16)} \qquad (F.41a)$$

Since y_n is periodic as shown in (F.5), r_s may be written in this alternate form:

$$r_s\ \ = \frac{1}{P} \sum_{n=0}^{P-1} y_n\, y_{n+s}\ =\ <y_n\, y_{n+s}>_1 \qquad (F.41b)$$

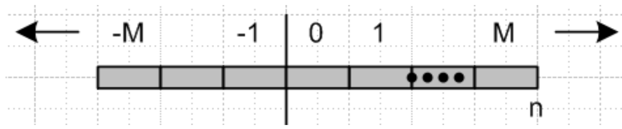
<u>Proof</u>: For large N, we can replace the sum endpoints by -N = -MP and +N = MP ≈ (M+1)P where M is also large. The sum then has (2M+1)P terms, so

$$r_s\ \equiv\ \lim_{M\to\infty} [\frac{1}{(2M+1)}\frac{1}{P} \sum_{n=-MP}^{(M+1)P} y_n\, y_{n+s}\ ]\ .$$

We now let n = IP + n' where n' takes values 0,1..P-1 and I = Int(n/P). Then the single sum above can be written,

$$\sum_{n=-MP}^{(M+1)P} y_n y_{n+s} = \sum_{I=-M}^{M} \sum_{n'=0}^{P-1} y_{IP+n'} y_{IP+n'+s}$$

where this drawing shows how the sum is now a double sum where I denotes segments containing P points, and n' counts the points in each segment,



We then process this sum using the periodicity (F.5) to get

$$= \sum_{I=-M}^{M} \sum_{n'=0}^{P-1} y_{n'} y_{n'+s} = (\sum_{I=-M}^{M} 1) (\sum_{n=0}^{P-1} y_n y_{n+s}) = (2M+1) (\sum_{n=0}^{P-1} y_n y_{n+s}) .$$

Inserting the expression into r_s, we get this alternate form for r_s

$$r_s = \lim_{M\to\infty} [\frac{1}{(2M+1)} \frac{1}{P} \{ (2M+1) (\sum_{n=0}^{P-1} y_n y_{n+s}) \}] = \frac{1}{P} \sum_{n=0}^{P-1} y_n y_{n+s} \qquad \textbf{QED}$$

From now on we assume that the y_m are real. Back in (F.25) we made the assumption that

$$\begin{array}{lll} \langle y_m y_n \rangle = \alpha & \text{for } m \neq n & s < | m-n | \\ \langle y_m y_n \rangle = \beta & \text{for } m = n & \end{array} \qquad \textit{(F.25)}$$

where <...> were ensemble averages. Here we are going to make a completely different assumption, and this assumption applies to a single sequence:

$$\begin{array}{lll} \langle y_m y_n \rangle_1 = \alpha & \text{for } m \neq n & s < | m-n | \\ \langle y_m y_n \rangle_1 = \beta & \text{for } m = n & \end{array} \qquad \text{(F.42)}$$

Comment: If we imagine drawing each sequence of an ensemble as a row in a set of rows

...	*	*	y_n	*	y_{n+2}	...	sequence #1
...	*	*	y'_n	*	y'_{n+2}	...	sequence #2
...	*	*	y''_n	*	y''_{n+2}	...	sequence #3

then any ensemble average like $\langle y_n y_{n+2} \rangle$ is a **vertical average** through the ensemble, whereas an average like $\langle y_n y_{n+2} \rangle_1$ is a **horizontal average** across one particular sequence row and $\langle y_n y_{n+2} \rangle_1$ never depends on n as (F.41) shows. These two averages are unrelated and (F.42) being true does not imply that (F.25) is true. As an example, one might take 10 infinite sequences each of which satisfies (F.42) and put them down as a set of rows, and then each row is shifted horizontally some amount to cause a 0 to be in column n. For the resulting 10 row ensemble, one would find that $\langle y_n^2 \rangle = 0$ for column n, whereas $\langle y_n^2 \rangle_1 = \beta \neq$

0 . This contrived situation, however, would be a violation of "stationarity" as discussed in (35.10), and in fact for a very large ensemble of our infinite pulse trains we will argue below that in fact $<y_n y_{n+2}> = <y_n y_{n+2}>_1$, but we put that argument on a back burner for the moment and maintain the distinction between <..> and $<..>_1$.

Since we require r_s for arbitrary s in order to compute R"(z), we extend (F.42) in this manner

$$<y_m y_n>_1 = \alpha \qquad \text{for } m \neq n + NP \qquad N = \text{any integer}$$
$$<y_m y_n>_1 = \beta \qquad \text{for } m = n + NP \qquad (F.43)$$

which is to say, for s being any integer whatsoever,

$$<y_n\, y_{n+s}>_1 = \alpha \qquad s \neq NP$$
$$<y_n\, y_{n+s}>_1 = \beta \qquad s = NP \quad . \qquad (F.44)$$

The reason of course is that $\{y_m\}$ is periodic with period P,

$$y_{m+NP} = y_m \qquad \text{for any integer N} \qquad \textit{(F.5)}$$

so we must have for example $<y_n\, y_{n+P}>_1 = <y_n\, y_n>_1 = \beta$.

Thus we have arrived at our characterization of the autocorrelation sequence r_s for our specific infinite periodic sequence with the assumption (F.44),

$$r_s = <y_n\, y_{n+s}>_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer} \qquad (F.45)$$

Our next step is to compute its Z transform R"(z),

$$R''(z) = \sum_{n=-\infty}^{\infty} r_n z^{-n} = \beta \sum_{n=NP} z^{-n} + \alpha \sum_{n \neq NP} z^{-n}$$
$$= \beta \sum_{n=NP} z^{-n} + \alpha \left(\sum_{n=-\infty}^{\infty} z^{-n} - \sum_{n=NP} z^{-n} \right)$$
$$= (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n} \quad . \qquad (F.46)$$

The first sum is over n = 0, ±P, ±2P and so on. We can replace summation index n by index N,

$$\sum_{n=NP} z^{-n} = \sum_{N=-\infty}^{\infty} z^{-NP} = \sum_{n=-\infty}^{\infty} z^{-nP} \quad . \qquad (F.47)$$

From (24.1) we know that z lies on the unit circle in the z-plane and is related to ω by

$$z = e^{i\omega T_1} \tag{24.1}$$

where T_1 is the duration of a pulse of the pulse train. We then have

$$\sum_{n=NP} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-nP} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 nP} . \tag{F.48}$$

According to (A.31),

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \tag{A.31}$$

so setting $k = -\omega T_1 P$ we get

$$\sum_{n=NP} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(-\omega T_1 P - 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P + 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) \tag{F.49}$$

where we use $\delta(x) = \delta(-x)$ and in the last step take $m \rightarrow -m$.

Meanwhile, our other sum of interest in (F.46) is this one,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-n} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 n}$$

which is just the previous sum without the P. Thus,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \tag{F.50}$$

Inserting (F.48) and (F.49) into (F.46) gives

$$R''(z) = (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n}$$

$$= (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (\beta-\alpha)\, (T_1 P)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/[T_1 P]) + \alpha (T_1)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/T_1)$$

$$= (2\pi/T_1) \{ (\beta-\alpha)(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/[T_1 P]) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/T_1) \}$$

$$= \omega_1 \{ (\beta-\alpha)(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \} \qquad (F.51)$$

and this concludes our calculation of the Z Transform R"(z) of the autocorrelation sequence r_s.

It only remains to install this into the Z Transform Wiener-Khintchine relation (34.14a) which says

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z) \qquad \textit{(34.14a)}$$

so then

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \omega_1 \{ (\beta-\alpha)\frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \}$$

$$= \mathcal{P}_{pulse}(\omega)\, \omega_1 \sum_{m=-\infty}^{\infty} [(\beta-\alpha) \frac{1}{P} \delta(\omega - \omega_1 m/P) + \alpha\, \delta(\omega - \omega_1 m)] \qquad (F.52c)$$

We may compare this to the ensemble result of the section (c) above,

$$<\mathcal{P}(\omega)> = \mathcal{P}_{pulse}(\omega)\, \omega_1 \sum_{m=-\infty}^{\infty} [(\beta-\alpha) \frac{1}{P} \delta(\omega - \omega_1 m/P) + \alpha\, \delta(\omega - \omega_1 m)] \quad . \qquad \textit{(F.33c)}$$

The expressions are exactly the same ! However, the meaning of the symbols α and β is not the same according to the definitions given above in (F.25) and (F.42).

Why are the expressions the same? This goes back to the general discussion of Section 35 (h) where it was shown that, for infinitely long pulse trains, $<y_m y_n>_1 = <y_m y_n>$ (35.24) and $<\mathcal{P}(\omega)> = \mathcal{P}(\omega)$ (35.27) and all pulse trains in the ensemble are statistical pulse trains having the same statistics and having the same $\mathcal{P}(\omega)$. Thus, our result (F.52c) above *had* to come out the same as (F.33c). Moreover, the quantities α and β are in fact the same values in the two cases.

Since the expressions have the exact same form, we can rewrite (F.52c) in the same alternate ways that (F.33c) was written:

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\, \frac{1}{P} [(\beta-\alpha) + \alpha\, 2\pi\, \delta_6(\omega T_1, \frac{P-1}{2})] \qquad (F.52a)$$

$$= \mathcal{P}_{pulse}(\omega)\, \omega_1 \frac{1}{P} [(\beta-\alpha) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) + \alpha \sum_{N=-\infty}^{\infty} \delta(\omega - \omega_1 N)\, P] \qquad // \; m = NP$$

$$= \mathcal{P}_{pulse}(\omega)\, \omega_1 [(\beta-\alpha) \frac{1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m)] \qquad (F.52b)$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\delta(\omega - \omega_1 m/P)\ +\ \alpha\ \delta(\omega - \omega_1 m)\ ] \qquad \text{(F.52c)}$$

$$= \mathcal{P}_{pulse}(\omega)\ \omega_1\ [\ (\beta-\alpha)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta-\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]$$

(F.52d)

(f) Summary and an Example: The MLS Sequence

Fact : For an infinite statistical sequence made of repeated subsequences of length P : (F.54)
if the following is found to be true,

$$\langle y_m y_n \rangle_1 = \alpha \qquad \text{for } m \neq n + NP \qquad N = \text{any integer}$$
$$\langle y_m y_n \rangle_1 = \beta \qquad \text{for } m = n + NP \qquad \textit{(F.43)}$$

then the spectral power density is given by : (this is one of several forms shown in (F.52)

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\delta(\omega - \omega_1 m/P)\ +\ \alpha\ \delta(\omega - \omega_1 m)\ ] \qquad \textit{(F.52c)}$$

The parameters α and β can be interpreted as elements of the autocorrelation sequence

$$r_s = \langle y_n\, y_{n+s} \rangle_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer} \qquad \textit{(F.45)}$$

The form shown in (F.52d) is

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1\ [\ (\beta-\alpha)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P)\ +\ \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m)\ +\ \{\ (\beta-\alpha)\ /P + \alpha\ \}\ \delta(\omega)\ ]$$

which has the following graphical representation (drawn for P = 4),

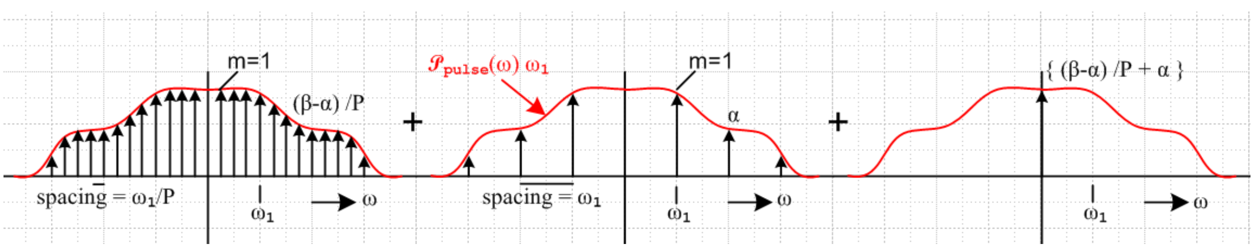


Fig F.2

Each vertical arrow represents a spectral δ line. The height of the arrow is the value of the red envelope curve times the quantity shown. The red curve $\mathcal{P}_{pulse}(\omega)\ \omega_1$ will in general have an infinite extent, an example being $\mathcal{P}_{pulse}(\omega)\ \omega_1 = \text{sinc}^2(\omega T_1/2) = \text{sinc}^2(\pi\omega/\omega_1)$ for a box shaped pulse as in (9.2).

As $P \to \infty$, we showed in section (d) the spectrum (F.52d) becomes,

$$\mathcal{P}_{\mathbf{pulse}}(\omega)\,\omega_1 \left[(\beta-\alpha) \;+\; \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) \;+ \alpha\, \delta(\omega) \right]$$

Regarding the three images of Fig F.2, we see that

- the left set of dense arrows coalesces into the continuous function $(\beta-\alpha)\,\mathcal{P}_{\mathbf{pulse}}(\omega)\,\omega_1$
- the middle set of arrows stays exactly the same
- the amplitude of the DC line becomes α

Once a particular $\mathcal{P}_{\mathbf{pulse}}(\omega)$ is specified, some of the spectral lines may be quenched. For the box pulse

$$\mathcal{P}_{\mathbf{pulse}}(\omega)\,\omega_1 = \mathrm{sinc}^2(\omega T_1/2) = \mathrm{sinc}^2(\pi\omega/\omega_1) \qquad (36.1)$$

lines are quenched when $\omega = N\omega_1$ for $N = \pm1,\pm2$ .... In this case, all the lines of the central image go away and the corresponding lines in the left image also vanish, this being every P^{th} line in that image:

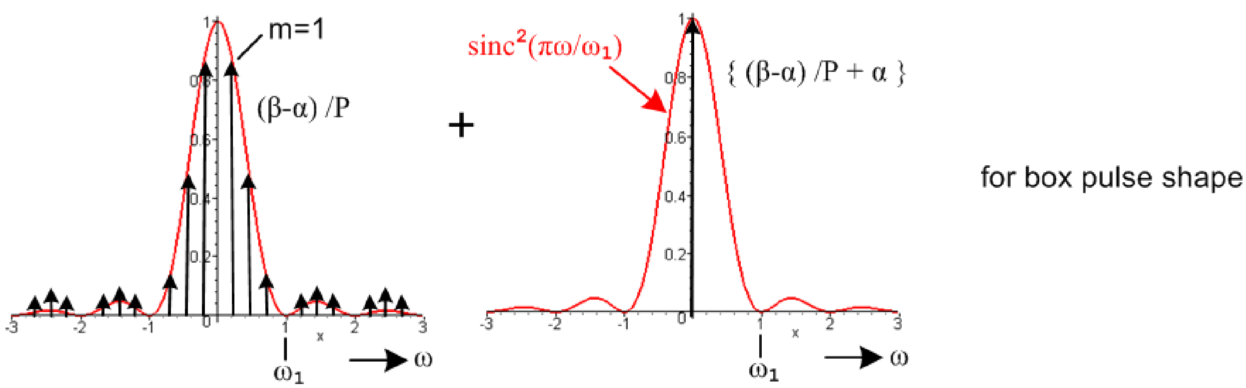


Fig F.3

Example: The MLS Sequence

A Maximum Length Sequence (MLS) (Lucht, *Polynomial Multipliers...*) which is created by a shift register generator has the property (F.44), specifically,

$$\begin{aligned} \langle y_n\, y_{n+s}\rangle_1 &= \alpha = (1/4)(1 + 1/P) && s \neq NP \\ \langle y_n\, y_{n+s}\rangle_1 &= \beta = (1/2)(1 + 1/P) && s = NP \end{aligned} \qquad (F.44)$$

where P must be one of the special values $P = 2^k-1$ for $k = 1,2,3$.... Therefore, by Fact (F.54) the spectrum $\mathcal{P}(\omega)$ of an MLS sequence is given by

$$\mathcal{P}(\omega) \;=\; \mathcal{P}_{\mathbf{pulse}}(\omega)\;\omega_1 \sum_{m=-\infty}^{\infty} \left[(\beta-\alpha)\, \frac{1}{P}\,\delta(\omega - \omega_1 m/P) \;+\; \alpha\, \delta(\omega - \omega_1 m) \right] . \qquad (F.52)$$

$$= [\mathcal{P}_{\mathbf{pulse}}(\omega)\,\omega_1]\;(1/4)(1 + 1/P) \sum_{m=-\infty}^{\infty} \left[\frac{1}{P}\,\delta(\omega - \omega_1 m/P) \;+\; \delta(\omega - \omega_1 m) \right] . \qquad (F.55)$$

(g) Results for an A,B repeated sequence

This subject is treated in Section 34 (c) using a "brute force" approach and a "Fourier Series" approach. Here we duplicate the main results of that section by applying our general formulas to a sequence where the repeated subsequence is just P = 2, $\{y_0,y_1\} = \{A,B\}$, and we allow A and B to be complex. The fact that the results here agree with Section 34 lends some confidence to all three methods of computation.

Our general expressions for X(ω) and $\mathscr{P}(\omega)$ are,

$$X(\omega) = X_{pulse}(\omega)\, \omega_1\, (1/P)\, Y_P''(z) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \tag{F.12}$$

$$\mathscr{P}(\omega) = \mathscr{P}_{pulse}(\omega)\, \omega_1\, (1/P)^2 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)\ |\, Y_P''(z)\, |^2 \tag{F.23}$$

$$\text{where} \quad Y_P''(z) \equiv \sum_{n=0}^{P-1} y_n\, e^{-i\omega n T_1} \quad . \tag{F.8}$$

For a repeated A,B sequence P = 2 we have

$$Y_P''(z) = A + B\, e^{-i\omega T_1}$$

$$|\, Y_P''(z)\, |^2 = |\, A + B\, e^{-i\omega T_1}\, |^2 = |A|^2 + |B|^2 + 2\, \mathrm{Re}\{A^*B\, e^{-i\omega T_1}\}$$

Then,

$$X(\omega)_{AB} = X_{pulse}(\omega)\, \omega_1\, (1/2)\ [A + B\, e^{-i\omega T_1}] \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega)_{AB} = \mathscr{P}_{pulse}(\omega)\, \omega_1\ (1/4)\, [\ |A|^2 + |B|^2 + 2\, \mathrm{Re}\{A^*B\, e^{-i\omega T_1}\}\ ] \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/2) \ .$$

We can slide the square-bracketed factors inside the sum and then use

$$\omega T_1 = (m\omega_1/2)T_1 = \pi\, m(\omega_1/2\pi)T_1 = \pi m \qquad => \qquad e^{-i\omega T_1} = (-1)^m$$

and the results simplify to

$$X(\omega)_{AB} = X_{pulse}(\omega)\, \omega_1\, (1/2) \sum_{m=-\infty}^{\infty} [A + B\, (-1)^m]\, \delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega)_{AB} = \mathscr{P}_{pulse}(\omega)\, \omega_1\ (1/4) \sum_{m=-\infty}^{\infty} [\ |A|^2 + |B|^2 + 2\, \mathrm{Re}(A^*B)\, (-1)^m\ ]\, \delta(\omega - m\omega_1/2) \quad .$$

These results agree with (34.18) and (34.20), see just below Fig 34.1 .

For a **box pulse** of height 1 and width T_1 we know that

$$X_{pulse}(\omega) = T_1 \,\text{sinc}(\omega T_1/2) = (1/\omega_1)\, 2\pi \,\text{sinc}(\omega T_1/2) \qquad (9.2)$$

$$\mathscr{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (1/\omega_1)\, \text{sinc}^2(\omega T_1/2) \;. \qquad (33.24)$$

Then for a square wave with alternating A,B amplitudes we get

$$X(\omega)_{AB} = 2\pi \,\text{sinc}(\omega T_1/2)\, (1/2) \sum_{m=-\infty}^{\infty} [A + B\,(-1)^m]\, \delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega)_{AB} = \text{sinc}^2(\omega T_1/2)\, (1/4) \sum_{m=-\infty}^{\infty} [\, |A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,(-1)^m \,]\, \delta(\omega - m\omega_1/2) \;.$$

When the sinc functions are moved inside the sum, $\omega T_1/2 = \pi m/2$, so

$$X(\omega)_{AB} = 2\pi\, (1/2) \sum_{m=-\infty}^{\infty} [A + B\,(-1)^m]\, \text{sinc}(\pi m/2)\, \delta(\omega - m\omega_1/2)$$

$$\mathscr{P}(\omega)_{AB} = (1/4) \sum_{m=-\infty}^{\infty} [\, |A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,(-1)^m \,]\, \text{sinc}^2(\pi m/2)\delta(\omega - m\omega_1/2) \;.$$

But

$$\text{sinc}(\pi m/2) = (2/\pi m)\sin(\pi m/2) \quad \begin{aligned} &= 1 \text{ for } m = 0 \\ &= 0 \text{ for } m \text{ even} \\ &= (2/\pi m)\,(-1)^{(m-1)/2} \text{ for } m \text{ odd} \end{aligned}$$

so we get

$$X(\omega)_{AB} = 2\pi\, (1/2) \sum_{m=odd} [A - B\,(-1)^m]\, (2/\pi m)\, (-1)^{(m-1)/2}\, \delta(\omega - m\omega_1/2) + 2\pi\, (1/2)\, [A+B]\, \delta(\omega)$$

$$\mathscr{P}(\omega)_{AB} = (1/4) \sum_{m=odd} [\, |A|^2 + |B|^2 + 2\,\text{Re}(A^*B)\,(-1)^m \,]\, (2/\pi m)^2\, (-1)^{(m-1)}\, \delta(\omega - m\omega_1/2)$$
$$+\, \omega_1\, (1/4)\, [\, |A|^2 + |B|^2 + 2\,\text{Re}(A^*B) \,]\, \delta(\omega)$$

or

$$X(\omega)_{AB} = 2 \sum_{m=odd} [A - B]\, (1/m)\, (-1)^{(m-1)/2}\, \delta(\omega - m\omega_1/2) + \pi\, [A+B]\, \delta(\omega)$$

$$\mathscr{P}(\omega)_{AB} = (1/\pi^2) \sum_{m=odd} [\, |A|^2 + |B|^2 - 2\,\text{Re}(A^*B) \,]\, (1/m)^2\, \delta(\omega - m\omega_1/2)$$
$$+\, \omega_1\, (1/4)\, [\, |A|^2 + |B|^2 + 2\,\text{Re}(A^*B) \,]\, \delta(\omega)$$

For a standard-issue square wave with B = - A we have

$$[A - B] = 2A \qquad [\ |A|^2 + |B|^2 - 2\,\mathrm{Re}(A^*B)\ ] = 4\,|A|^2$$
$$[A + B] = 0 \qquad [\ |A|^2 + |B|^2 + 2\,\mathrm{Re}(A^*B)\ ] = 0$$

and therefore

$$X(\omega)_{A,-A} = 4A \sum_{m=odd} (1/m)\,(-1)^{(m-1)/2}\,\delta(\omega - m\omega_1/2)$$

$$\mathcal{P}(\omega)_{A,-A} = 4A^2\,(1/\pi^2) \sum_{m=odd} (1/m)^2\,\delta(\omega - m\omega_1/2)$$

in agreement with (34.23).

Appendix G: Random Variables, Probability Theory and Pulse Train Amplitudes

This subject always seems a bit slippery. One can find a discussion like the one presented below in every book on probability theory, but here we wish perhaps to put our own "spin" on the subject with wordy comments not always appearing in texts.

(a) What is a Random Variable ? Part I

A random variable only exists and has meaning in a certain context. The context is that there is some **experiment** that is performed many times, and the random variable is associated with a particular **outcome** of that experiment. An experiment is "something one does" or "something that is done" and an outcome is "something one measures after the experiment is done". A simple example is experiment = **roll one die**, outcome = number facing up after the die is rolled = random variable n. The set of all possible outcomes is called the **sample space**, often indicated by Ω (element = ω) or S (element = s). For the **die** experiment, we have Ω = {1,2,3,4,5,6} as the sample space. Since this is a set, various concepts regarding sets can be applied. A set we know has subsets. Each subset of the sample space has a peculiar name, each subset is called an **event**. In the die experiment, a possible event would be {2,4,6} which event is that the die rolled an even number. Another event would be {6} which event is that the die rolled a 6. For this experiment, the sample space is **discrete**.

So far we have not defined "random variable" but have said it is "associated with" the outcome of some experiment. If we assume that the experimental outcome of interest takes real numerical values, then the "variable" of the random variable *is* that outcome. The "random" part of random variable means that the variable does not take a fixed value, but instead takes a range of values as the experiment is repeated. After the experiment is done many times, one can compute a probability distribution for the outcome, so there is always a probability distribution associated with a random variable.

A common wrong impression inferred from the word "random" in this context is that the probability distribution must be "flat". For the die experiment, a flat distribution would mean that the probability of each face-up value was P_n = 1/6. This is not what random means in this context. It just means that there exists some distribution of outcome values. A better phrase might be "statistical variable" or "probabilistic variable". If a die is loaded so that P_n is different for the 6 faces, n is still a random variable.

If the distribution of outcomes happens to be a Gaussian (normal) distribution, then one might refer to the variable as a "Gaussian random variable".

Therefore, for an experiment which has a real numerical outcome, the "random variable" *is* that outcome and there exists a non-trivial probability distribution *for* that outcome which can be determined by doing the experiment many times. In order to say one has a "random variable", one must be able to state both the experiment that gets done, and the particular outcome of that experiment that is of interest.

Next, consider experiment = one spins a "**spinner**". This is a traditional (in probability texts) physical object that one might think of as a horizontal Lazy Susan having an arrow from center to some point on the rim, or a Roulette Wheel with some point marked on the rim, or just a metal arrow you spin and drop on a table top. The outcome of this experiment is the direction in which the arrow points in range (0, 2π) perhaps clockwise relative to North. In this example, the sample space is all real numbers in the interval (0,2π). For this experiment, the sample space is a **continuous** set which one could regard as the limit of a discrete set as the number of elements increases.

We shall now assume that in the die experiment the die was a "loaded" die of some sort, and that in the spinner experiment, the arrow was perhaps influenced by magnetic fields or by bad bearings on the Lazy Susan. If we do each experiment many times and each time we observe and write down the outcome, we can then "bin" these outcomes and create a distribution by dividing each bin count by the total number of experiments done (see section (f) below) . Here are possible distributions for our two experiments.

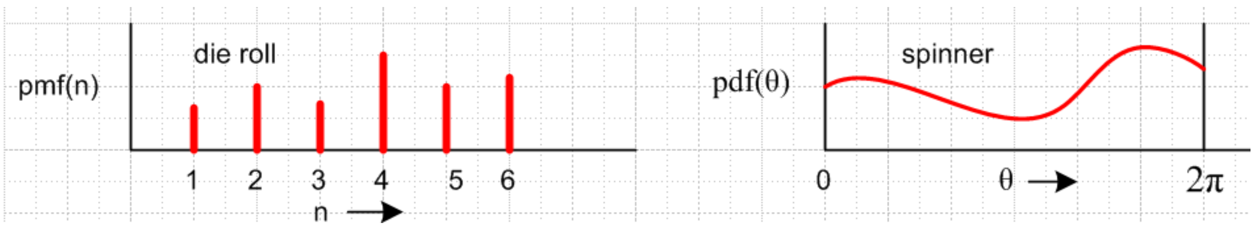


Fig G.1

For the spinner, the distribution is continuous and is called a **probability density function** or **pdf**. For the die, the distribution is discrete as is called a **probability mass function** or **pmf** . One could in fact write down a pdf for the left picture assuming n was a continuous variable in this manner,

$$\mathrm{pdf}(n) = \Sigma_{i=1}^{6} \ \mathrm{pmf}(n)\, \delta(i-n) \ . \qquad \text{(G.1)}$$

Both a pmf and pdf must sum to one, since the probability of *some* outcome is always 1,

$$\Sigma_{k=1}^{6} \mathrm{pmf}(k) = 1 \qquad\qquad \int_0^{2\pi} \mathrm{pdf}(x)dx = 1.$$

Notice that the above equations are consistent in that

$$1 = \int_0^{2\pi} \mathrm{pdf}(n)dn = \int_0^{2\pi} [\ \Sigma_{i=1}^{6} \ \mathrm{pmf}(n)\, \delta(i-n)]dn = \Sigma_{i=1}^{6} \mathrm{pmf}(n) \int_0^{2\pi} \delta(i-n)]dn$$

$$= \Sigma_{i=1}^{6} \mathrm{pmf}(n) = 1 \ .$$

The motivation for these names pdf and pmf comes from physics. One can have a continuous distribution of mass in some compressible fluid, say, where it would be called a mass **density** ρ. But in the physics of idealized point particles of **mass** m_i, the mass is congealed at specific points in space. For a set of particles of mass m_i at spatial locations $\mathbf{r}_i$ one could still write a mass density $\rho(\mathbf{r})$ as

$$\rho(\mathbf{r}) = \Sigma_i \, m_i \, \delta(\mathbf{r}-\mathbf{r}_i)$$

and one sees the analogy with (G.1) above.

Here then we have seen two "random variables" in action. One, n, is the discrete outcome of a die roll, the other θ is the continuous outcome of a spinner spin. Each of these is a real parameter and for each there exists an associated probability distribution, as plotted above. Thus, each of these parameters fulfills the requirements of our opening definition of a "random variable".

(b) What is a Random Variable ? Part II: the Capital Letter Notation

We shall continue to examine various experiments to see what new concepts appear.

Consider experiment = **coin toss** with sample space S = {heads,tails} = possible outcomes. The new feature here is that the outcomes are not real numbers and therefore don't fit our requirement that a random variable be a real number. This is easily remedied by assigning real numbers to each outcome, such as heads → 1 and tails → 0. If s is an element of the sample space S = {heads,tails}, then we can define a little function h = H(s) such that 1 = H(heads) and 0 = H(tails). In general this function is a mapping from the sample space S to the real numbers, so we can say H:S→R. In this situation, we can think of h as being a "random variable" as defined above. It is a real parameter and it has an associated probability distribution which in this case is a pmf since S is discrete. For a "loaded" coin we might have

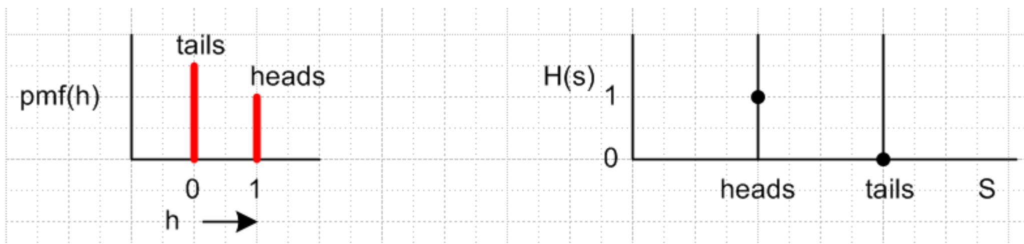


Fig G.2

We are now going to slightly alter our definition of a random variable so it is more precise. As a preliminary, forget probability and just consider some function y = f(x). In this equation, the object f is clearly a function, while the object y is a value that function can take. Thus, y and f are not the same thing. They are equal in the sense that y = f(x) for some x, but y and f belong to completely different classes of objects. Perhaps y is a real *variable* in R, whereas f is a *mapping* (function) from R to R. It is quite useful to have different symbols for y and f. If one were to write y = y(x), the reader would understand what was meant, but now y has two separate meanings (function, and value of function). To maintain precision, it is better that these symbols be different. One choice is to represent the function f(x) as Y(x), so the function name is now Y, while y is a value the function Y can take. Then y = Y(x) and things are clear.

Now we return to h = H(s). H is a function or mapping from S to R, whereas h is a real number. We would not in general say that h and H are the same object. For the coin toss experiment, it is really the function H that is the random variable, whereas h is a value that random variable can take. In our previous experiments with die and spinner, it happened that the function H was the identity function, so we didn't "see it" in the discussion. But still for the spinner, we would write θ = Θ(θ) and, as we set things up, if an arrow position s is in the range (0,2π), then we happen to have Θ(θ) = θ and Θ = 1. The function Θ would have been more visible had we perhaps taken arrow position in degrees and put θ in radians.

We shall now state a more complete definition of a random variable:

First one must have a specific experiment E in mind which gets run many times, and on each run of the experiment some outcome S of interest takes a value s_E in a sample space S_E of possible outcomes for S. The **random variable** $X_{E,S}$ is a function which maps the sample space S_E to the real axis, $X_{E,S} : S_E \rightarrow R$ so that $X_{E,S}(s_E) = x_{E,S}$, where then $x_{E,S}$ is the real value which the random variable $X_{E,S}$ takes for a particular outcome s_E in S_E. After the experiment is run many times, one can determine a probability distribution $P_{E,S}(x_{E,S})$ for this outcome S of this experiment E.

Here we have shown annoying subscripts just to stress that we have a specific experiment E and a specific outcome set S in mind. We now restate our definition of random variable without these annoying subscripts:

First one must have a specific experiment in mind which gets run many times, and on each run of the experiment some outcome of interest takes a value s in a sample space S of possible outcomes. The **random variable** X is a function which maps the sample space S to the real axis, $X : S \rightarrow R$ so that X(s) = x, where then x is the real value which the random variable X takes for a particular outcome s in S. After the experiment is run many times, one can determine a probability distribution P(x) for this outcome of this experiment.

This P(x) may or may not be a "flat" distribution. If one thinks of a flat distribution as being the definition of a random distribution, one arrives at the famous statement that a random variable is neither a variable (it is a function) nor is it random (distribution can be non-flat).

What different name might one use for "random variable"? Perhaps a "probabilistic function" which maps the possibly non-numeric outcomes of an experiment to a real parameter, which parameter has some non-trivial probability distribution which can be observed by doing the experiment many times. The problem with this phrase is that it does not focus on that parameter which is the main item of interest, the variable of random variable. So we just learn to live with the phrase "random variable".

For some general experiment, the outcomes are likely to be non-numerical in nature, and a function like H(s) will be required to map the outcomes onto the real number axis. For a discrete sample space, we might represent this situation as follows:

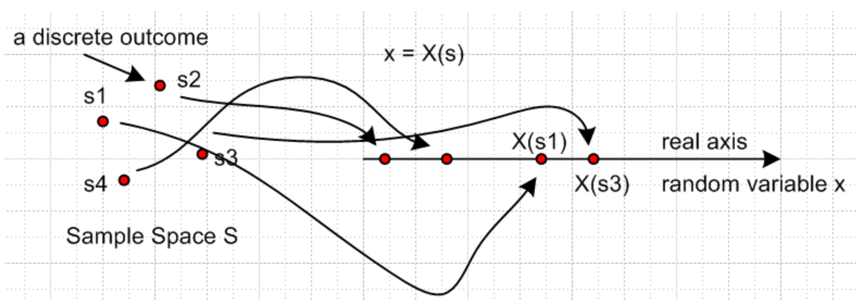


Fig G.3

In this case, an event which is any subset of the outcomes in the sample space will map into some set of points on the real axis. For a continuous sample space, the picture is a little different,

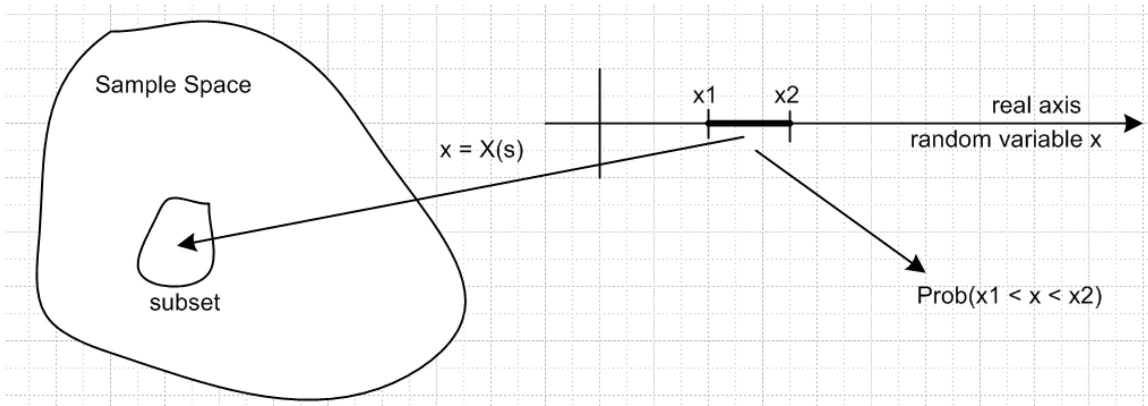


Fig G.4

The new feature here is a certain continuity requirement: every interval of the random variable x (like the interval shown) must map back into some subset of the sample space. Since subsets are events, this means that every interval of the x axis must map back to some well-defined event. Going the other way, not every event (subset) maps into an interval of the x axis. The continuity idea is that if two points are close together on the real x axis, then they must be close together in the sample space. This in turn means that the sample space has to have some kind of metric to allow a notion of distance between two points. Furthermore, as the subset is expanded, the interval it maps to cannot becomes smaller!

Having now discussed random variables, outcomes, experiments, sample spaces, and events, we are in a good position to review some general probability theory.

(c) Basic Probability Theory

In the above discussion, if we have random variables A,B,C.... , we can write a **joint probability** density function in this manner

$$\text{pdf}(A=a, B=b, C=c.....) \tag{G.2}$$

where now we don't distinguish whether the various sample spaces are continuous or discrete, we just write anything as a pdf (with the understanding of (G.1) above). The meaning here is that pdf(...) is the probability that random variable A has value a, while at the same time (the same experiment) random variable B has value b, and so on. For the continuous case, pdf(...) da db dc... is the same probability but for the range da of a and db of b, etc.

If all these random variables are **statistically independent** (such as A = number of dust particles on your pillow and B = temperature at some location on Pluto), this joint probability distribution factors,

$$\text{pdf}(A=a,B=b,C=c.....) = \text{pdf}(A=a) * \text{pdf}(B=b) * \text{pdf}(C=c) \; .... \tag{G.3a}$$

For two random variables A and B, one would have

$$\text{pdf}(X=x,Y=y) = \text{pdf}(X=x) * \text{pdf}(Y=y) \; . \tag{G.4a}$$

We shall often use the word independent to mean statistically independent.

We now adopt some **shorthand notations**:

$$p(a,b,c....) \equiv pdf(A{=}a,B{=}b,C{=}c.....) = pdf_{ABC...}(a,b,c...) = p_{ABC..}(a,b,c..) . \qquad (G.5)$$

When one sees p(2, -4, 0. 4...) *one must remember* that the arguments correspond to values of specific random variables, and if things become unclear, one must revert to the fuller notation. One trick is to use a parameter name that reminds the reader of the random variable name, such as a for A: p(a) = P(A=a).

In the shorthand notation we have

$$p(a,b,c...) = p(a)p(b)p(c)..... \qquad \text{// N statistically independent random variables} \qquad (G.3b)$$

$$p(x,y) = p(x)p(y) \qquad \text{// 2 statistically independent random variables} \qquad (G.4b)$$

Digression: One way to understand the concept of statistical independence is by the use of conditional probabilities. Consider:

$p_X(x \mid y)$ = probability *just for X* that X = x *given that* Y = y

which we compare to

$p_{XY}(x,y)$ = probability *for X and Y* that X = x and Y = y .

The connection is given by,

$$p_{XY}(x,y) = p_X(x \mid y)\, p_Y(y) .$$

If X and Y are independent, then $p_X(x \mid y)$ has no dependence on y, X knows nothing about Y, and in this case we have $p_X(x \mid y) = p_X(x)$. This then yields the factored form

$$p_{XY}(x,y) = p_X(x)\, p_Y(y) .$$

For three variables we can define

$p_X(x \mid y, z)$ = probability *just for X* that X = x *given that* Y = y and Z = z

and then (start on the right end when reading this)

$$p_{XYZ}(x,y,z) = p_X(x \mid y, z)\, p_Y(y \mid z)\, p_Z(z) .$$

If X,Y,Z are statistically independent, then certainly X knows nothing about Y and Z, and Y knows nothing about Z, so then $p_X(x \mid y, z) = p_X(x)$ and $p_Y(y \mid z) = p_Y(y)$ and then

$$p_{XYZ}(x,y,z) = p_X(x)\, p_Y(y)\, p_Z(z)$$

giving the factored form. Often this discussion appears with the $\cap$ symbol replace our commas, and one can verify the various claims with Venn diagrams. We shall not digress more on this subject and shall take the factored form as our definition of statistical independence.

We return now to

$$p_{XY}(x,y) = p_X(x)p_Y(y) \qquad \text{// 2 statistically independent random variables} \qquad \textit{(G.4b)}$$

If one regards $p_{XY}(x,y)$ as a function $f_y(x)$ for various fixed values of y, (G.4b) says that the *shape* of this function is not influenced by the values of y, only the *overall scale* of $f_y(x)$ is affected by y.

After we define the correlation measure corr(X,Y) below, we will see that (G.4b) being true implies that corr(X,Y) = 0 which means X and Y are **uncorrelated**. This arrangement works only one way:

$$\text{X,Y statistically independent} \qquad \Rightarrow \qquad \text{X,Y uncorrelated.}$$

It is easy to find examples where X,Y are uncorrelated but are not independent, we shall look at an example below.

Any pdf is **normalized** to 1 since the probability of all possible outcomes (mapped from sample spaces to the random variables) is 1. Thus,

$$\int\int .... \ p(x,y....)\, dx\, dy... \ = 1 \qquad \text{or} \qquad \Sigma_{x,y....}\ p(x,y....) \ = 1 \ . \qquad (G.6)$$

An example is that $\int p(x)dx = 1$ or $\Sigma_x p(x) = 1$. If x is discrete and y is continuous, $\Sigma_x \int p(x,y)dy = 1$, but we won't bother to show all such "mixed" cases below.

In general, if p(a,b,c...) is some N^{th} order joint probability, one can find lower order probabilities by summing over some of the variables:

Fact: To obtain a lower-order joint probability from a higher-order one,

$$p(x,y,z...) = \Sigma'_{a,b,c...} \ p(a,b,c.....) \qquad (G.7)$$

Here {x,y,z...} is some subset of {a,b,c....} and the notation Σ' means that we sum over all variables in {a,b,c.... } *except* those in the subset { x,y,z...} . One could state (G.7) in a set notation this way

$$p(S') = \Sigma_{S-S'}\ p(S) \qquad \text{where } S' \subset S \qquad (G.8)$$

Proof: The proof is merely the observation that if we are only interested in {x,y,z...}, we go ahead and let all the other variables {a,b,c....} – {x,y,z...} take all possible values and we add up the probability of each of these cases to get its contribution to p(x,y,z...). This is just a case of adding probabilities of outcomes to get a total probability of interest.

Examples:

$p(x) = \Sigma_y \, p(x,y)$

$p(x) = \Sigma_{y,z} \, p(x,y,z)$

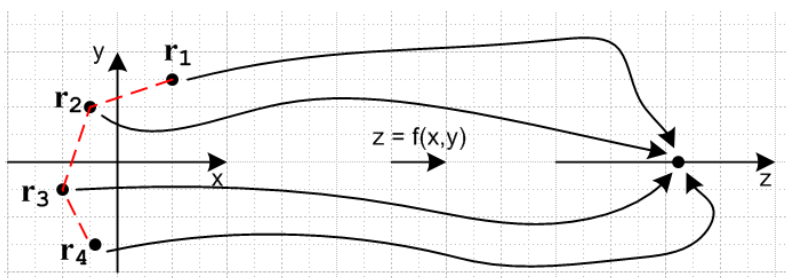
$p(x,y) = \Sigma_z \, p(x,y,z)$

$p(x,y) = \Sigma_{a,b,c... \neq x,y} \, p(a,b,c......)$ (G.9)

Before we can continue our little presentation, we need to state and prove an important theorem about random variables which are functions of other random variables. To this end, we must first digress on a set of math Lemmas.

Some Math Lemmas

Consider z = f(x,y) = f(**r**) where **r** = (x,y). Here f is a "function" which means it is a single-valued function which means under the mapping f: $R^2 \rightarrow R$, every vector **r** in the domain lands in some unique location in the range. It is possible and in fact likely that f will be a many-to-one function, meaning for a given z in the range, there might be several $\mathbf{r}_i$ in the domain such that $z = f(\mathbf{r}_i)$. If the domain is discrete, then so is the range. We might have this situation:



domain D_f range R_f Fig G.5

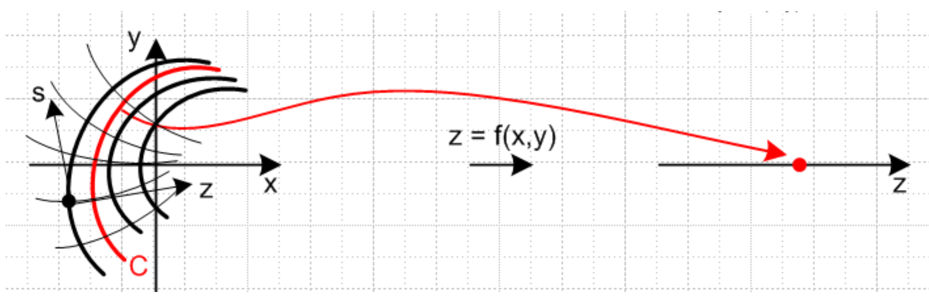
We could then talk about $\Sigma_{x,y}$ being the sum of all points **r** = (x,y) in the domain such that f(x,y) = z for some z in the range. We can exhaust all points in the range R_f by doing $z_i = f(\mathbf{r}_i)$ and letting $\mathbf{r}_i$ exhaust all (x,y) in the domain D_f. This is how we discover the extent of the range R_f.

We now make this claim, where we have in mind some unseen quantity being acted upon by both sides of the equation (symbol ∍ means "such that")

$$\sum_{z \in R_f} \left[\sum_{x,y \ni f(x,y) = z} \right] = \sum_{x,y \in D_f} \qquad \text{(G.10)}$$

Proof: If on the left we first sum over the points (x,y) corresponding to z according to z = f(x,y), and *then* we sum over all z in the range of f, our sum includes every point (x,y) in D_f exactly once. The double sum is simply a certain ordering of the total sum shown on the right. The same point (x,y) cannot show up twice in the double sum for two different values of z, because if it did, f(x,y) would map that point into those two z different values, but f is supposedly single-valued. Nor can any point (x,y) be omitted in the double sum on the left because the range R_f for z was created by exhausting all points (x,y) in D_f, so any (x,y) in D_f corresponds to some z in the range R_f.

Now for a continuous domain D_f, we might have



domain D_f range R_f Fig G.6

where now an entire continuous curve C (red) in the domain maps to some z in R_f. The equation of the red curve C is z = f(x,y) for some fixed value of z. The heavy curves including this red one are curves of constant z. We can imagine some other function s = g(x,y) whose curves of constant s form an orthogonal curvilinear coordinate system (right angles at any point) with the curves of z = f(x,y) = constant. The parameter s would then vary along each constant-z curve, marking off points along the curve. The analogous statement to the discrete statement made above is this (only a claim, we shall not prove it)

$$\int_{z \in R_f} dz \left[\int_{C(z)} ds \ J \right] = \iint_{x,y \in D_f} dxdy \qquad \text{(G.11)}$$

where J is the Jacobian between coordinates (x,y) and (z,s). This is the same Jacobian idea that appears in taking (x,y) to polar coordinates (r,θ) where dxdy = rdrdθ with J = r. We shall not pursue this Jacobian matter further other than to claim it is possible to find a g(x,y) that works and that certain technical issues arise concerning the reasonableness of the function f(x,y).

Comment: In order for the Jacobian to exist and be well-behaved, function f(x,y) must be continuous and differentiable (C^1) in both variables, *and* the mapping between (x,y) and (z,s) must be essentially one- to- one (invertible) so that given (z,s) one can compute (x,y) and vice versa.

Example: Suppose $z = f(x,y) = 2x^3 + 3y^2$ and we are interested in the curve C(z=2). This curve C is the intersection of the surfaces $z = 2x^3 + 3y^2$ (red) and z = 2 (gray), as illustrated here:

```
p1 := plot3d(2*x^3+3*y^2,  x=-2..2,y=-2..2, color=red):
p2 := plot3d(2,  x=-2..2,y=-2..2, color = gray,style = patchnogrid):
display(p1,p2);
```

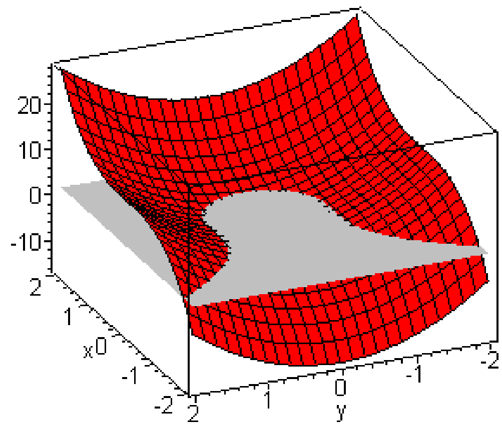


Fig G.7

If we had $z = f(x,y,w)$ so Z is a function of *three* random variables X,Y and W, then our two results above would become

$$\sum_{z \in R_f} \left[\sum_{x,y,w \ni f(x,y,w) = z} \right] = \sum_{x,y,w \in D_f}$$

$$\int_{z \in R_f} dz \left[\int_{S(z)} dS \; J \right] = \iint_{x,y,w \in D_f} dxdydw \qquad \text{(G.12)}$$

The first line seems straightforward, but the second is more complicated. Inside the square bracket we now have dS being a differential patch of area, and S(z) is a 2D surface in the (x,y,z) space on which z is a constant according to $z = f(x,y,w)$. For example, in spherical coordinates we write $r = x^2+y^2+z^2$ and a surface of constant r is a spherical shell. Now we have to imagine two other coordinates $s_1 = g(x,y,w)$ and $s_2 = g(x,y,w)$ so that (z,s_1,s_2) form an orthogonal coordinate system and then the new J is the Jacobian between (x,y,w) and (z,s_1,s_2).

In a more economical notation, and changing to $z = f(a,b,c)$ we can write (G.10) and (G.11) as

$$\Sigma_z [\Sigma^z_{x,y}] = \Sigma_{x,y} \qquad \text{(G.13a)}$$

$$\int dz \, [\int_{C(z)} ds \, J] = \int \int da \, db \qquad \text{(G.13b)}$$

and then for (G.9),

$$\Sigma_z [\Sigma^z_{a,b,c}] = \Sigma_{a,b,c} \qquad \text{(G.14a)}$$

$$\int dz \, [\int_{S(z)} dS \, J] = \int \int \int da \, db \, dc \qquad \text{(G.14b)}$$

where $\Sigma^z_{a,b,c}$ indicates that the sum is restricted to those (a,b,c) values for which $f(a,b,c) = z$.

Hopefully it is clear how this general idea can be extended to Z = f(A,B,C,D....) where random variable Z is a function of some arbitrary number N of random variables A,B,C,D... .

Fact: One can organize a total sum/integral over the space of N random variables in this manner:

$$\Sigma_z [\Sigma^z_{a,b,c...}] = \Sigma_{a,b,c...} \qquad \text{(G.15a)}$$

$$\int dz \; [\int_{S(z)} dS \; J] = \int\int....\int da\, db\, dc ...... \qquad \text{(G.15b)}$$

In the first line, Σ^z means the sum is constrained to be only over those a,b,c... such that f(a,b,c...) = z. In the second line, S(z) is an N-1 dimensional surface located within the N dimensional space of (a,b,c....), which surface is defined by f(a,b,c....) = z .

With these Lemmas out of the way, we can now resume our basic probability theory review.

Fact: Any reasonable real function of random variables is a random variable. (G.16)

Proof: First consider Z = f(X,Y), where X and Y are random variables. If x is an allowed value of X, and y of Y, then the allowed values of Z will be z = f(x,y). Since x and y must be real, and since f is a real function, the allowed values of z are real, one of the requirements for Z to be a random variable. At our level of rigor, it only remains to find the probability distribution associated with Z. We claim this is given by,

$$p(z) = \Sigma^z_{x,y} \; p(x,y) \qquad \text{(G.17a)}$$

$$p(z) = \int_{C(z)} ds \; J \; p(x(s,z),y(s,z)) = \int_{C(z)} J \; p(x,y) \; . \qquad \text{(G.17b)}$$

Looking at (G.17a), for a *given value* of z, in z = f(x,y) only certain x and y values are possible. Thus, only these values of x and y appearing in p(x,y) can contribute to p(z). Other values of x and y will contribute to p(z') for some other z'. We can check normalization as follows using (G.13a)

$$\Sigma_z \, p(z) = \Sigma_z\{ [\Sigma^z_{x,y}] \, p(x,y)\} = \{\Sigma_z \, [\Sigma^z_{x,y}]\} \, p(x,y) = \Sigma_{x,y} \; p(x,y) = 1 \; .$$

What is the sample space for Z ? We can select one according to the following plan. Let R_A and R_B be the total sets of values of parameters a and b. Then $R_Z = f(R_A,R_B)$ treated as an equation of sets tells us the set R_Z. We could then just select the sample space for Z to be $S_Z = R_Z$ which would be a numerical sample space.

More generally, for Z = f(A,B,C....) with N arguments, the distribution for Z is given by

$$p(z) = \Sigma^z_{a,b,c...} \; p(a,b,c,....) \qquad \text{(G.18a)}$$

$$p(z) = \int_{C(z)} dS \; J \; p(a,b,c....) \qquad a = a(z, s_1, s_2..... s_{N-1}), \text{ etc.} \qquad \text{(G.18b)}$$

The function f must be C^1 in all its arguments (as noted above) and must be such that the Jacobian J is well behaved (finite and non-zero) over the entire regions of interest in both spaces (a,b,c...) and $(z,s_1,s_2...)$. This is why (G.16) contains the word "reasonable". **QED**

Definition : The **expected value** of a random variable X is given by

$$E(X) \equiv \int x\, p(x)\, dx \quad \text{or} \quad E(X) \equiv \Sigma_x\, x\, p(x) \qquad = \textbf{the mean}\text{, often called } \mu_X \qquad \text{(G.19)}$$

Comment 1: The expected value is sometimes called the expectation or the expectation value. In quantum mechanics the phrase expectation value predominates, but elsewhere it is the expected value. In quantum mechanics, all physical observables are random variables (position, momentum, energy, etc) except in quantum states which are eigenstates of the observable's quantum operator, in which case the observable takes a fixed value.

Comment 2: In (G.19) above for the mean (and generally below), we could write $E(X) \equiv \Sigma_i\, x_i\, p(x_i)$ where the x_i are the possible values that random variable X can take. But in this section we use $E(X) \equiv \Sigma_x\, x\, p(x)$ where then x itself represents the values X can take. This notation puts the summation form on a little more equal footing with the integral form $\int x\, p(x)\, dx$.

Fact: If Z = f(X,Y), the expected value of Z is given by (G.20)

$$E(Z) = \Sigma_z\, z\, p(z) = \Sigma_{x,y}\, f(x,y)\, p(x,y) \qquad \text{discrete}$$

or

$$E(Z) = \int z\, p(z)\, dz = \int\int f(x,y)\, p(x,y)\, dx\, dy\,. \qquad \text{continuous}$$

Proof: For the discrete case

$$E(z) = \Sigma_z\, z\, p(z) \qquad \text{// by the definition of an expected value of a random variable}$$

$$= \Sigma_z\, z\, [\, \Sigma^z{}_{x,y}\, p(x,y)] \qquad \text{// by (G.17a)}$$

$$= \Sigma_z\, [\, \Sigma^z{}_{x,y}]\, f(x,y)\, p(x,y) \qquad \text{// move the x,y sum to the left, and replace z = f(x,y)}$$

$$= \Sigma_{x,y}\, f(x,y)\, p(x,y) \qquad \text{// by (G.13a)}$$

For the continuous case

$$E(Z) = \int z\, p(z)\, dz = \int dz\, z\, [\int_{C(z)} ds\, J\, p(x,y)\,] \qquad \text{// by (G.17b)}$$

$$= \int dz\, [\int_{C(z)} ds\, J\,]\, f(x,y)\, p(x,y) \qquad \text{// move } \int ds \text{ to the left, replace z = f(x,y)}$$

$$= \int\int dxdy\;\; f(x,y)\, p(x,y) \qquad \text{// by (G.13b)}$$

QED

We now generalize this fact to obtain:

Theorem: If Q = f(X,Y,Z....) where X,Y,Z... are random variables, and if f is a reasonable real valued function, then

(1) Q is a random variable and

(2) $E(f(X,Y,Z....)) = \Sigma_{x,y,z...} f(x,y,z....)\, p(x,y,z....)$

$$E(f(X,Y,Z....)) = \int\int\int ...dxdydz....\ f(x,y,z....)\, p(x,y,z....) \qquad (G.21)$$

<u>Proof</u>: This is a straightforward generalization (G.20) based on (G.18) and (G.15). One just mimics the proof of (G.20).

Special cases:

$$E(f(X)) = \Sigma_x f(x)\, p(x)$$
$$E(f(X)) = \int dx\, f(x)\, p(x) \qquad (G.22)$$

$$E(f(X,Y)) = \Sigma_{x,y} f(x,y)\, p(x,y)$$
$$E(f(X,Y)) = \int\int dx\, dy\, f(x,y)\, p(x,y) \qquad (G.23)$$

Next we define the **covariance** of X and Y, and evaluate it using (G.23),

$$\mathrm{cov}(X,Y) \equiv E((X-\mu_x)(Y-\mu_y)) = \Sigma_{x,y} (x-\mu_x)(y-\mu_y)\, p(x,y)$$
$$\text{or} \int\int (x-\mu_x)(y-\mu_y)\, p(x,y)\, dx\, dy \qquad (G.24a)$$

Notice that

$$\mathrm{cov}(X,Y) \equiv E(XY) - \mu_x E(Y) - \mu_y E(x) + \mu_x\mu_y = E(XY) - \mu_x\mu_y - \mu_y\mu_x + \mu_x\mu_y = E(XY) - \mu_x\mu_y$$

so we have this alternate method of computing covariance

$$\mathrm{cov}(X,Y) = E(XY) - \mu_x\mu_y \quad . \qquad (G.24b)$$

In passing, notice that cov(X,-Y) = – cov(X,Y) since E(XY) negates as does μ_x.

The **variance** of X is the covariance of X with itself and is the square of the **standard deviation** σ(X),

$$\mathrm{var}(X) \equiv [\sigma(X)]^2 = \mathrm{cov}(X,X) = E((X-\mu_x)(X-\mu_x)) = E((X-\mu_x)^2)$$
$$= \Sigma_x (x-\mu_x)^2\, p(x) \quad \text{or} \quad \int (x-\mu_x)^2\, p(x)\, dx \qquad (G.25)$$

where we use (G.22) with $f(X) = (X-\mu_x)^2$.

Finally, the **correlation** of X and Y is the covariance normalized by the two standard deviations,

$$\mathrm{corr}(X,Y) \equiv \mathrm{cov}(X,Y) / [\sigma(X)\,\sigma(Y)\,] \ . \qquad \text{(G.26)}$$

The most correlated that X and Y could possibly be occurs when X = Y in which case corr(X,Y) = +1. The most anticorrelated they can be occurs when X = -Y in which case corr(X,Y) = -1. For general X and Y, we have $-1 \le \mathrm{corr}(X,Y) \le 1$.

Example of uncorrelated but dependent. Let X = Z and $Y = Z^2$. Let p(z) = an even function of z, like a Gaussian. Assume for Z that $\mu_z = 0$. It seems fairly clear that X and Y are not independent, they know a lot about each other, they are dependent and $p(z,z^2) \ne p(z)p(z^2)$. Nevertheless, the correlation is 0 as we now show: (cov = 0 => corr = 0)

$$\mathrm{cov}(X,Y) = E(XY) - \mu_x\mu_Y \ = E(Z^3) - \mu(Z)\,\mu(Z^2) \ = E(Z^3) = \int_{-\infty}^{\infty} dz\, p(z)\, z^3 \ = \ 0 \ .$$

Just to get the following delta forms on the table, we now present an alternate derivation (G.25). One can take the limit of a joint distribution of random variables as two of the variables become the same. For example, for continuous and discrete (where we now use the alternate indexed notation with x_i),

$$\lim_{X\to Y} p(x,y) = p(x)\delta(x-y) \qquad\qquad \lim_{X\to Y} p(x_i,y_i) = p(x_i)\delta_{i,j} \ . \qquad \text{(G.27)}$$

The reason is that if X and Y are the same random variable, they cannot take different values. If X,Y and Z are all the same variable, then we would have (dropping the limit notation)

$$p(x,y,z) = p(x)\delta(x-y)\delta(x-z) \qquad \text{or} \qquad p(x_i,y_j,z_k) = p(x_i)\,\delta_{i,j}\,\delta_{i,k} \qquad \text{(G.28)}$$

Note that $\delta(x-y)\delta(x-z) = \delta(x-y)\delta(y-z) = \delta(x-z)\delta(y-z)$ and $\delta_{i,j}\,\delta_{i,k} = \delta_{i,j}\,\delta_{j,k} = \delta_{i,k}\,\delta_{j,k}$.

We can apply (G.27) to obtain an alternate evaluation of the variance,

$$\mathrm{var}(X) \equiv [\sigma(X)]^2 = \lim_{X\to Y} \mathrm{cov}(X,Y) \ = \lim_{X\to Y} [\ \int\int (x-\mu_x)\,(y-\mu_Y)\,p(x,y)\,dx\,dy\ ] \qquad \text{(G.29)}$$

$$= \int\int (x-\mu_x)\,(y-\mu_x)\,p(x)\delta(x-y)\,dx\,dy$$

$$= \int (x-\mu_x)^2\,p(x)\,dx \qquad \text{or} \qquad \Sigma_i\,(x_i-\mu_x)^2\,p(x_i) \ / \ \Sigma_x\,(x-\mu_x)^2\,p(x)$$

which is the same as (G.25).

Notice that the variance could in theory vanish if $p(x) = \delta(x - x_1)$ so that the entire pdf is concentrated at a single value and then $\mu_x = x_1$:

$$\mathrm{var}(X) = \int (x-\mu_x)^2\, p(x)\, dx = \int (x-\mu_x)^2\, \delta(x - x_1)\, dx = (x_1-\mu_x)^2 = (x_1- x_1)^2 = 0 = \sigma(X) .$$

But then X is not a random variable since it's value is precisely determined as x_1. A pdf of this form is not very interesting and normally one has $\sigma(X) > 0$.

Fact: $\mathrm{var}(X) = \sigma^2(X) = E(X^2) - \{E(X)\}^2 = E(X^2) - \mu_x^2$ (G.30)

Proof: $E(X^2) - \{E(X)\}^2 = \int x^2 p(x)\, dx - \{\int x\, p(x)\, dx\}^2 = \int x^2 p(x)\, dx - \mu_x^2$

$$\mathrm{var}(X) = \int (x-\mu_x)^2\, p(x)\, dx = \int x^2 p(x)\, dx - 2\mu_x \int x\, p(x)\, dx + \mu_x^2 \int p(x)\, dx$$

$$= \int x^2 p(x)\, dx - 2\mu_x\, \mu_x + \mu_x^2\, 1 = \int x^2 p(x)\, dx - \mu_x^2 .$$ **QED**

Fact: If X and Y are statistically independent random variables, then (G.31)

(a) $p(x,y) = p(x)p(y)$
(b) $E(XY) = E(X)E(Y)$
(c) $\mathrm{cov}(X,Y) = 0$
(d) $\mathrm{corr}(X,Y) = 0$
(e) X and Y are uncorrelated

Proof:

(a) follows from our definition of statistical independence (G.3a) or (G.3b).

(b) $E(XY) = \int\int x\, y\, p(x,y)\, dx\, dy = \int\int x\, y\, p(x)p(y)\, dx\, dy = [\int x\, p(x)\, dx]\, [\int y\, p(y)\, dy] = E(X)E(Y)$.

(c) $\mathrm{cov}(X,Y) \equiv E(XY) - \mu_x\mu_y = E(X)E(Y) - \mu_x\mu_y = \mu_x\mu_y - \mu_x\mu_y = 0$

(d) $\mathrm{corr}(X,Y) \equiv \mathrm{cov}(X,Y) / [\sigma(X)\, \sigma(Y)] = 0 / [\sigma(X)\, \sigma(Y)] = 0$

(e) This is the same as (d), just stated in words **QED**

Fact: If {X,Y} are independent, and if {Y,Z} are independent, {X,Z} may be dependent and therefore be correlated. (G.32)

Proof: Knowing that $p(x,y) = p(x)p(y)$ and $p(y,z) = p(y)p(z)$ tells us nothing about $p(x,z)$. It is easy to think of trivial examples of this fact. Maybe X = Z so $\mathrm{corr}(X,Z) = \mathrm{var}(X)/ [\sigma(X)]^2 = 1 \neq 0$.

Fact: If {X,Y} and {Y,Z} and {X,Z} are all independent pairs, X Y and Z might not be independent. (G.33)

Proof: Knowing about the uncorrelated pairs says nothing about p(x,y,z). That is to say, knowing that p(x,y) = p(x)p(y) and p(y,z) = p(y)p(z) does not imply that p(x,y,z) = p(x)p(y)p(z).

This concludes our brief review of probability theory, and we now continue in our examination of experiments associated with random variables.

(d) Ensemble Experiments

We consider now a new kind of "experiment". We acquire N dice which we shall refer to as an **ensemble** of dice. We assume they are all "loaded" *differently*, perhaps with implanted weights. For each die, we perform the single-roll experiment described above for which the outcome lies in the sample space S = {1,2,3,4,5,6}. After doing these N experiments, the N dice are left lying on the green felt of a craps table, each in its final experimental state. We then survey all N dice and take note of each one's face-up number, and from that data we construct a distribution. Since these dice are all slightly different, the distribution will likely differ from that shown earlier in Fig G.1. Perhaps we get this:

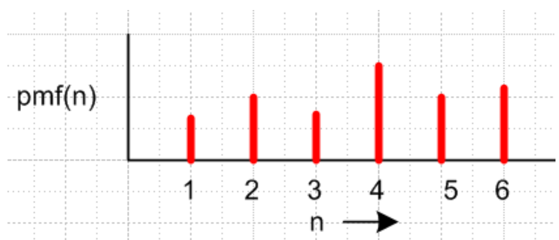


Fig G.8

As a computer algorithm, here is this new "ensemble experiment":

Acquire an ensemble of N dice (perhaps each is loaded differently)
For each die in the ensemble, carry out the single die experiment with outcome in S = {1,2,3,4,5,6}.
When the experiments are done, survey the resulting data and construct a distribution.

If the dice were identical, then this ensemble experiment would be the same as just sequentially rolling the same die N times and writing down the outcome of each roll. But the main point is that we assume the dice are *not* all the same.

Now we carry out an analogous ensemble experiment :

Acquire an ensemble of N babies (they are likely all different)
For each baby in the ensemble, carry out some experiment with outcome in some sample space S.
When the experiments are done, survey the resulting data and construct a distribution.

The experiment we have in mind is simply to let each baby grow up into an adult and then we treat some parameter of that adult as our random variable. Adults generally don't have face-up numbers, but they do have other "parameters" such as mass, height, and number of children. For each adult, the "experiment" was "growing up", analogous to rolling one die in our previous ensemble experiment. It is convenient to let nature do these experiments for us, so in practice we just assemble some group of adults into our ensemble, and then we are left with just the last item above :

When the experiments are done, survey the resulting data and construct a distribution.

Here for example we carry out three ensemble experiments with the same ensemble of people. Each ensemble experiment uses a different sample space: weight, height, number of children. and each deals with a different random variable: W, H and N. For each experiment we obtain a distribution as shown:

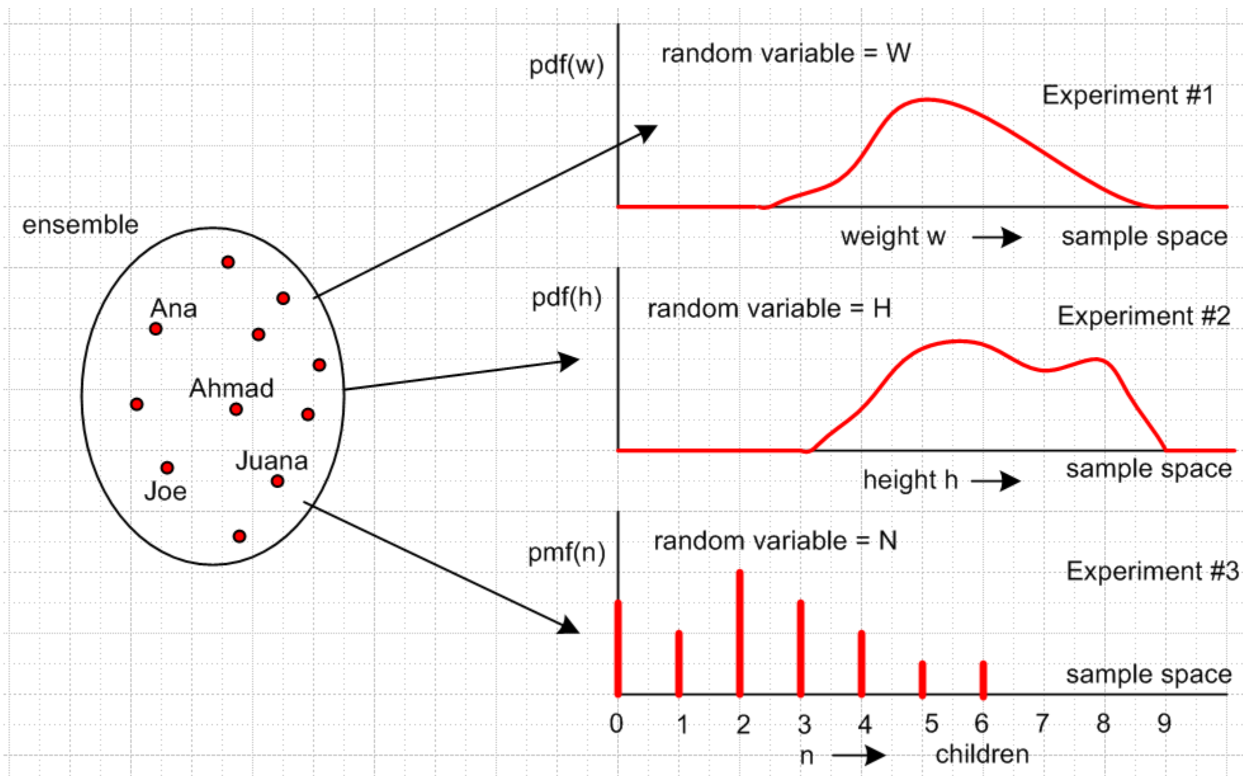


Fig G.9

In all three experiments, the outcomes are real numbers, so there is no need for any functions to translate from non-numerical outcomes to real numbers, as we required in for the {heads,tails} sample space. These experiments are all analyzing historical data, nor current "chance events" like rolling dice.

In the above scenario, sometimes the ensemble of people is called "the sample space" which is then a totally different meaning of that phrase, so we won't use it.

(e) Experiments rolling two dice at the same time

In the next set of experiments, we roll two differently weighted dice at the same time and examine particular outcomes (that is to say, we examine particular random variables, each taking real values in its numerical sample space). These two dice are not only weighted differently, but the embedded weights are magnets, which cause the two die to interact with each other during a roll. Here we get to apply various facts from the brief review of probability theory of section (c) above.

In **Experiment #1** we just look at the face up number of the X die and that number is x. The sample space as usual is S = {1,2,3,4,5,6} and there will be some $pmf_{\mathbf{X}}(x)$. Review: In this notation, the subscript indicates the random variable X, and the argument x is a value that random variable can take. Another notation is pmf(X = x), and the most compact notation is that of (G.5) which is just p(x) where one must remember what the random variable is. It is suggested by the letter used as argument: $p(x)= pmf_{\mathbf{X}}(x)$ = pmf(X = x).

In **Experiment #2** we do this for the Y die. It has the same sample space, but a different $p(y) = pmf_{\mathbf{Y}}(y)$.

In **Experiment #3** we look at z = x+y with its random variable Z = X + Y. The sum of two random variables is a random variable according to (G.16) or (G.21). The numerical sample space for this third experiment is {2,3..., 11,12}, since there are no other possible outcomes for the sum of two face-up numbers on two dice. According to (G.17a), the probability of Z taking value z in this sample space is given by $pmf_{\mathbf{Z}}(z) = \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ p_{\mathbf{XY}}(x,y)$, or in compact notation, $p(z) = \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ p(x,y)$, where the sum includes only those x,y values such that x + y = the z inside p(z) on the left. If the two die were differently weighted but contained no magnets, we would have $p(z) = \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ p_{\mathbf{X}}(x)p_{\mathbf{Y}}(y)$ since the two dice are "independent" (hence "uncorrelated") as in (G.31). If the two dice were identical, then $p(z) = \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ p_{\mathbf{X}}(x)p_{\mathbf{X}}(y)$ where the two pmf's are the same. If the two dice are "fair dice" (unloaded), then $pmf_{\mathbf{X}}(x) = p_{\mathbf{X}}(x) = 1/6$ for any x and then we find that $p(z) = \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ (1/6)(1/6) = (1/36)\ \Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ 1$. We then have a classic dice problem where we can enumerate the terms in the sum $\Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ 1$ as follows

z	$\Sigma^{\mathbf{z}}{}_{\mathbf{x,y}}\ 1$
2	1,1 = 1
3	1,2 + 2,1 = 2
4	1,3 + 3,1 + 2,2 = 3
...	
7	1,6 + 6,1 + 5,2 + 2,5 + 3,4 + 4,3 = 6
...	
12	6,6 = 1

Below, $pmf_{\mathbf{X}}(x)$ is for Experiment #1, and $pmf_{\mathbf{Z}}(z)$ for Experiment #3. In each experiment, we roll a pair of identical fair dice many times and obtain these distributions.

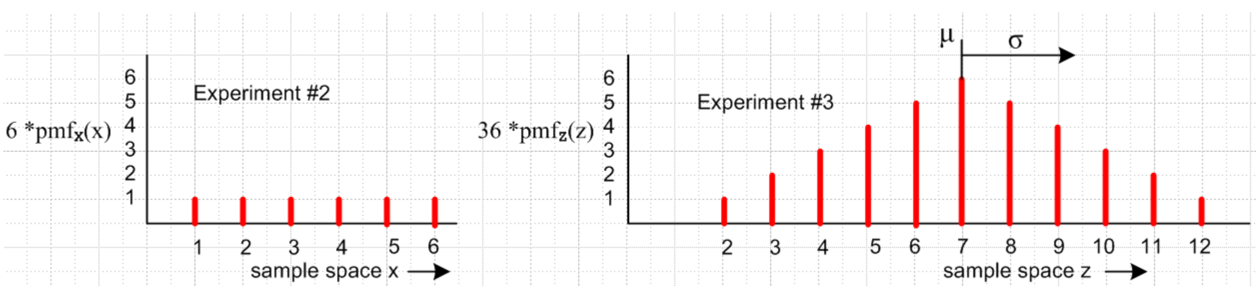


Fig G.10

We now do a few quick hand calculations to exercise some of our basic probability facts. Notice ahead of time that

$\Sigma_{\mathbf{x}}\ 1 = 6$
$\Sigma_{\mathbf{x}}\ x = (1 + 2 + 3 + 4 + 5 + 6) = 21$
$\Sigma_{\mathbf{x}}\ x^2 = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = 91$.

The expected value of Z for Experiment #3 is given by (G.23),

$$E(Z) = E(X+Y) = \Sigma_{\mathbf{x,y}}\ (x+y)\ p(x,y)\ .$$

In the case of Fig G.10 where $p(x,y) = [p(x)]^2 = (1/6)^2 = (1/36)$ we have

$$E(Z) = (1/36)\ \Sigma_{\mathbf{x,y}}(x+y) = (1/36)\ [\ (\Sigma_{\mathbf{x}}\ x)(\Sigma_{\mathbf{y}}1) + (\Sigma_{\mathbf{x}}\ 1)(\Sigma_{\mathbf{y}}y)\ ] = (1/18)\ (\Sigma_{\mathbf{x}}\ x)(\Sigma_{\mathbf{y}}1)$$

$$= (1/18)\ (21)\ (6) = (1/3)(21) = 7 = \mu_{\mathbf{Z}}$$

in agreement with the symmetric distribution on the right in Fig G.10. Next we compute,

$$E(Z^2) = \Sigma_{\mathbf{x,y}}\ (x+y)^2\ p(x,y) = (1/36)\ \Sigma_{\mathbf{x,y}}\ [\ x^2 + y^2 + 2xy\ ]$$

$$= (1/36)\ [\ 2\ (\Sigma_{\mathbf{x}}x^2)(\Sigma_{\mathbf{y}}1) + 2\ (\Sigma_{\mathbf{x}}x)^2\ ] = (1/18)\ [91*\ 6 + 21^2] = 329/6 = 54.83.$$

The variance and standard deviation are then given by (G.30).

$$\text{var}(Z) = E(Z^2) - \{E(Z)\}^2 = 329/6 - 49 = 35/6 = 5.833$$
$$\sigma(Z) = \sqrt{35/6} = 2.415 \qquad \text{(G.34)}$$

This last result seems in line with a visual inspection of Fig G.10.

Finally in **Experiment #4** we consider the outcome z = x*y = xy. Things are similar to the above, but now the notation $\Sigma^{\mathbf{z}}_{\mathbf{x,y}}$ means the sum over x,y values such that the *product* of x and y is z. In the context of (G.23), we now have f = xy whereas in experiment #3 we had f = x+y. We could of course study the situation with any reasonable function f. For f (x,y) = xy we can write:

$\text{pmf}_{\mathbf{Z}}(z) = \Sigma^{\mathbf{z}}_{\mathbf{x,y}}\ \text{pmf}_{\mathbf{XY}}(x,y)$	dice are different and weighted with magnets
$\text{pmf}_{\mathbf{Z}}(z) = \Sigma^{\mathbf{z}}_{\mathbf{x,y}}\ \text{pmf}_{\mathbf{X}}(x) * \text{pmf}_{\mathbf{Y}}(y)$	dice are different but no magnets
$\text{pmf}_{\mathbf{Z}}(z) = \Sigma^{\mathbf{z}}_{\mathbf{x,y}}\ \text{pmf}_{\mathbf{X}}(x) * \text{pmf}_{\mathbf{X}}(y)$	dice are identical
$\text{pmf}_{\mathbf{Z}}(z) = \Sigma^{\mathbf{z}}_{\mathbf{x,y}}\ (1/6) * (1/6) = (1/36)\ \Sigma^{\mathbf{z}}_{\mathbf{x,y}}\ 1$	dice are identical and "fair"

For this Experiment # 4 the sample space is all possible products S = {1, 2, 3......36} with certain values missing. For "identical and fair" we make our list using our new meaning of $\Sigma^{\mathbf{z}}_{\mathbf{x,y}}$:

z	$\Sigma^{z}_{x,y}$ 1
1	1,1 = 1
2	1,2 + 2,1 = 2
3	1,3 + 3,1 = 2
4	1,4 + 4,1 + 2,2 = 3
5	1,5 + 5,1 = 2
6	1,6 + 6,1 + 2,3 + 3,2 = 4
7	= 0
8	2,4 + 4,2 = 2
...	
36	6,6 = 1

We ask Maple to create $pmf_Z(z)$ for this Experiment #4:

```
for z from 1 to 36 do p(z) := 0 od:  # clear counters
for z from 1 to 36 do
  for x from 1 to 6 do
    for y from 1 to 6 do
      if z = x*y then p(z) := p(z) +1 fi;
    od;
  od;
od;
print(seq(p(z),z=1..36));
        1, 2, 2, 3, 2, 4, 0, 2, 1, 2, 0, 4, 0, 0, 2, 1, 0, 2, 0, 2, 0, 0, 0, 2, 1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 1
with(stats):with(stats[statplots]):
data1 := [seq(Weight(i..i+0.3,0.3*p(i)),i = 1..36)]:
histogram(data1, color=red,tickmarks = [10,5], view = [0..37,0..5]);
```

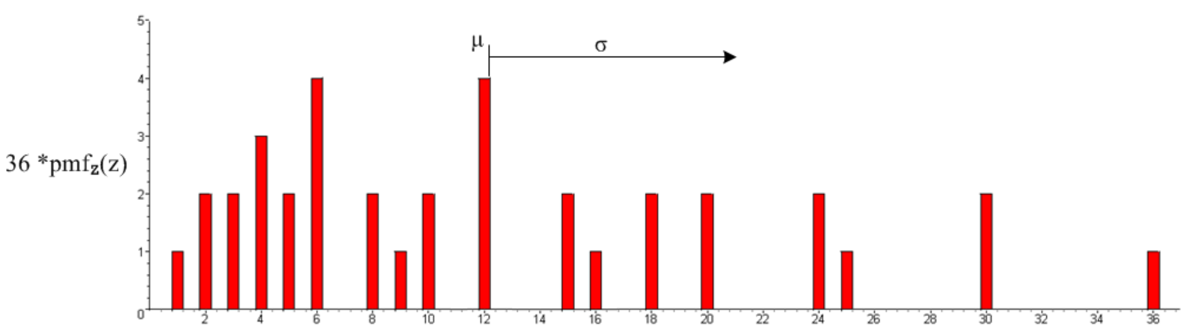


Fig G.11

We then repeat the calculations done for Experiment #3:

$E(Z) = E(XY) = \Sigma_{x,y}\,(xy)\,p(x,y) = (1/36)\,\Sigma_{x,y}\,(xy) = (1/36)[\,(\Sigma_x x)^2 = 21^2/36 = 49/4 = 12.25 = \mu_Z$

$E(Z^2) = \Sigma_{x,y}\,(xy)^2\,p(x,y) = (1/36)\,\Sigma_{x,y}\,x^2y^2 = (1/36)\,[\,(\Sigma_x x^2)^2] = (1/36)(91)^2 = 8281/36 = 230.03$

$var(Z) = E(Z^2) - [E(Z)]^2 = 8281/36 - (49/4)^2 = 11515/144 = 79.97$

$\sigma(Z) = 8.942$ // standard deviation (G.35)

(f) Experimental determination of discrete distribution functions

In practice, one has some list of experimental results (outcomes converted to real numbers if not already real numbers) and one puts the results into "bins" to obtain a distribution. Here we want to be more explicit about what this means.

For a **single variable** x which is the face-up value of a die, here is how we would compute p(x) after collecting data rolling the die J times :

```
restart;   J := 20: roll := rand(1..6):
x := seq( roll(),j=1..J);   # roll die J times
                    x := 4, 3, 4, 6, 5, 3, 6, 3, 2, 2, 2, 4, 4, 3, 3, 2, 1, 4, 4, 6
for n from 1 to 6 do bin[n] := 0 od:    # clear bins

for j from 1 to J do
  for n from 1 to 6 do
    if x[j] = n then bin[n] := bin[n] + 1 fi;
  od;
od;
for n from 1 to 6 do p[n] := bin[n]/J; od:
for n from 1 to 6 do printf("  p[%d] = %.3f ,",n,p[n]);od;

p[1] = .050 ,  p[2] = .200 ,  p[3] = .250 ,  p[4] = .300 ,  p[5] = .050 ,  p[6] = .150
```

which we could plot as a bar chart of the type shown in Fig G.1.

Symbolically we might write this as

$$p(n) = (1/J)\, \Sigma_{j=1}^{J} (\, x_j = n\,) \qquad n = 1,2,3,4,5,6\ . \tag{G.36}$$

The idea is that for each n, we count the number of times that ($x_j = n$) is true.

For **two variables**, things are a bit more complicated. Now we have a set of J pairs $\{x_i,y_i\}$ as our collection of data, so that

$$p(n,m) = (1/J)\, \Sigma_{j=1}^{J} (\, [x_i,y_i] = [n,m]\,) \qquad n,m = 1,2,3,4,5,6 \tag{G.37}$$

where [...] refers to an ordered sequence. Another way to write this would be

$$p(n,m) = (1/J)\, \Sigma_{j=1}^{J}\ [(n = x_i) \text{ and } (m = x_j)] \qquad n,m = 1,2,3,4,5,6 \tag{G.38}$$

Here is a sample Maple program to carry this out

```
> restart;   J := 20: roll := rand(1..6):
> x := seq( [roll(),roll()],j=1..J);   # roll two dice J times
x := [4, 3], [4, 6], [5, 3], [6, 3], [2, 2], [2, 4], [4, 3], [3, 2], [1, 4], [4, 6], [1, 1], [1, 2], [4, 2], [1, 3], [6, 3],
  [6, 4], [3, 6], [1, 6], [3, 1], [4, 6]
> for n from 1 to 6 do
    for m from 1 to 6 do
      cnt := 0;
      for j from 1 to J do
        if (x[j,1] = n) and (x[j,2] = m) then cnt := cnt + 1; fi
      od;
      p[n,m]:= cnt/J;
    od;
  od;
  for n from 1 to 6 do
      for m from 1 to 6 do printf(" p[%d,%d]=%.3f ,",n,m,p[n,m]);od;
      printf("\n");
  od;
 p[1,1]=.050 , p[1,2]=.050 , p[1,3]=.050 , p[1,4]=.050 , p[1,5]=0.000 , p[1,6]=.050 ,
 p[2,1]=0.000 , p[2,2]=.050 , p[2,3]=0.000 , p[2,4]=.050 , p[2,5]=0.000 , p[2,6]=0.000 ,
 p[3,1]=.050 , p[3,2]=.050 , p[3,3]=0.000 , p[3,4]=0.000 , p[3,5]=0.000 , p[3,6]=.050 ,
 p[4,1]=0.000 , p[4,2]=.050 , p[4,3]=.100 , p[4,4]=0.000 , p[4,5]=0.000 , p[4,6]=.150 ,
 p[5,1]=0.000 , p[5,2]=0.000 , p[5,3]=.050 , p[5,4]=0.000 , p[5,5]=0.000 , p[5,6]=0.000 ,
 p[6,1]=0.000 , p[6,2]=0.000 , p[6,3]=.100 , p[6,4]=.050 , p[6,5]=0.000 , p[6,6]=0.000 ,
```

The resulting p(n,m) can be visualized as a 3D bar chart

```
with(stats):with(stats[statplots]):
for n from 1 to 6 do
  data[n] := [seq(Weight(m..m+0.3,0.3*p[n,m]),m = 1..6)]:
od:
histogram(seq(data[n],n=1..6),color = red);
```

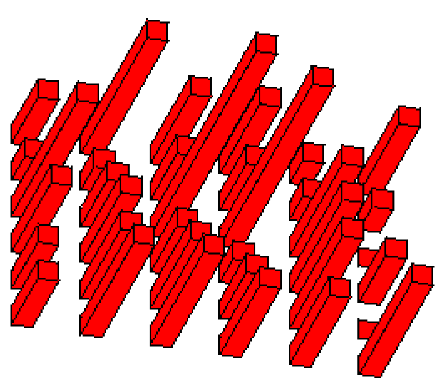


Fig G.12

A distribution function like p(a,b,c,d,e) is a 6D bar chart, not easy to display.

(g) Experiments with Sequences of Pulse Train Amplitudes

Our pulse trains have a set of amplitudes y_n which form a sequence. If the sequence has N elements, then a sequence can be written [y_1, y_2.....y_N]. When many pulse trains are generated in some line code, one finds that the values of y_n in position n of the sequence can vary, and that there is some probability distribution associated with the parameter y_n. Thus, the position n in the pulse train or sequence is associated with a random variable we must call Y_n which takes values y_n.

Here is the Experiment. We have an Apparatus which generates sequences of length N according to some set of rules implemented within the Apparatus (perhaps these are the rules for generating AMI linecode sequences, and the AMI encoder inputs random input streams). Every sequence it generates is "legal" according to its rules. Earlier we discussed an ensemble experiment in which M dice were rolled one at a time and were left on the craps table for study and from what we saw on the table, we were able to construct a distribution function. The ensemble was M dice. In our current **ensemble experiment**, we let the Apparatus crank away and generate an ensemble of M sequences each of length N, and these M sequences are left sitting on the same craps table for us to inspect. From this data we could then construct any desired statistical expected value or distribution function.

The notation Y_n leads to some possible confusion. When we had a random variable H which took values h, we might have enumerated the set of h values of the sample space as h_i. To maintain clarity, if we have random variable Y_n which takes values y_n, we should enumerate those values as $(y_n)_i$. Here, y_n is the name of a variable, just as h was the name of a variable. Thus, to write down a specific sequence "i" we really should say

$$[(y_1)_i, (y_2)_i .....(y_N)_i] \quad = \text{sequence "i"}$$

or (G.39)

$$[y_1^{(i)}, y_2^{(i)} ..... y_2^{(i)}] \quad = \text{sequence "i"}$$

where $(y_2)_i = y_2^{(i)}$ is some specific symbol value like "5". We used the latter notation when talking about an ensemble of pulse trains in Section 35.

In random variable discussions (like ours above in section (c)) one usually reads about "two random variables X and Y" with equations like these (written in discrete notation)

$\mu_x = E(X) = \Sigma_k x_k p_x(x_k) = (1/I) \Sigma_{i=1}^{I} x^{(i)} \equiv \langle x \rangle$
$\mu_y = E(Y) = \Sigma_k y_k p_y(y_k) = (1/I) \Sigma_{i=1}^{I} y^{(i)} \equiv \langle y \rangle$

$E(XY) = \Sigma_k\Sigma_j x_k y_j p_{xy}(x_k,y_j) = (1/I) \Sigma_{i=1}^{I} x^{(i)} y^{(i)} \equiv \langle xy \rangle$
$p_{xy}(x_k,y_j) = p_x(x_k) p_y(y_j)$ // independent
$E(XY) = E(X)E(Y)$. // independent

$cov(X,Y) = E([X - \mu_x] [Y - \mu_y]) = \Sigma_k\Sigma_j (x_k - \mu_x) (y_j - \mu_y) p_{xy}(x_k,y_j)$
$= (1/I) \Sigma_{i=1}^{I} (x^{(i)} - \mu_x) (y^{(i)} - \mu_y) = E(XY) - \mu_x \mu_y$

$var(X) \equiv \sigma_x^2 \equiv cov(X,X) = E(X^2) - \mu_x^2$

$corr(X,Y) = cov(X,Y) / [\sigma(X)\sigma(Y)$

For the pulse train application, we have a random variable Y_n for each position in a sequence or pulse train, so the idea is to think of $X = Y_n$ and $Y = Y_m$ and rewrite equations like those above in some notation that is unambiguous:

$\mu_{Ym} = E(Y_m) = \Sigma_k y_m^{(k)} p_{Ym}(y_m^{(k)}) = (1/I) \Sigma_{i=1}^{I} y_m^{(i)} = \langle y_m \rangle$
$\mu_{Yn} = E(Y_n) = \Sigma_k y_n^{(k)} p_{Yn}(y_n^{(k)}) = (1/I) \Sigma_{i=1}^{I} y_n^{(i)} = \langle y_n \rangle$

$E(Y_mY_n) = \Sigma_k\Sigma_j y_m^{(k)} y_n^{(j)} p_{YmYn}(y_m^{(k)}, y_n^{(j)}) = (1/I) \Sigma_{i=1}^{I} y_m^{(i)} y_n^{(i)} = \langle y_my_n \rangle$
$p_{YmYn}(y_m^{(k)}, y_n^{(j)}) = p_{Ym}(y_m^{(k)}) p_{Yn}(y_n^{(j)})$ // independent
$E(Y_mY_n) = E(Y_m) E(Y_n)$ or $\langle y_my_n \rangle = \langle y_m \rangle\langle y_n \rangle$ // independent

$cov(Y_m, Y_n) = E([Y_m-\mu_{Ym}] [Y_n-\mu_{Yn}]) = \Sigma_k\Sigma_j (y_m^{(k)} - \mu_{Ym}) (y_n^{(j)} - \mu_{Yn}) p_{YmYn}(y_m^{(k)}, y_n^{(j)})$
$= (1/I) \Sigma_{i=1}^{I} (y_m^{(i)} - \mu_{Ym}) (y_n^{(i)} - \mu_{Yn}) = E(Y_mY_n) - \mu_{Ym} \mu_{Yn}$

$var(Y_m) \equiv \sigma_{Ym}^2 \equiv cov(Y_m, Y_m) = E(Y_m^2) - \mu_{Ym}^2 = \langle y_m^2 \rangle - \mu_{Ym}^2$

$corr(Y_m, Y_n) = cov(Y_m, Y_n) / [\sigma(Y_m)\sigma(Y_n)]$

Detailed Summary of this Document

A brief summary may be found in the opening section before Chapter 1. The logical flow of the document is fairly complicated because so many interrelated topics are addressed, and because everything is derived from scratch for the reader to see. Hopefully this 12-page detailed summary can provide an intermediate level "map" of what is going on in the document.

Chapter 1: The Fourier Integral Transform and Related Topics (34 p)

Chapter 1 develops the basic theory of the Fourier Integral Transform and its Sine and Cosine cousins. This Chapter forms the underpinning of all subsequent Chapters. The Convolution Theorem receives special attention. The connection is made between the Laplace Transform and the "generalized" Fourier Transform applied to causal functions. Various "rules" are derived, and a connection is made between filter spectra and time-domain Green's Functions.

Section 1 introduces the Fourier Integral Transform, which in our notation appears as

$$X(\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-i\omega t} \qquad \text{projection = transform} \qquad (1.1)$$

$$x(t) = (1/2\pi) \int_{-\infty}^{\infty} d\omega\, X(\omega)\, e^{+i\omega t}, \qquad \text{expansion = inverse transform} \qquad (1.2)$$

and discusses restrictions on functions for which the transform applies and is meaningful. The notions of a "pulse" and a "pulse train" are mentioned in passing. The connection is made between the Fourier Integral Transform and the Fourier Sine and Cosine Integral Transforms. This relationship provides a method of using large published tables of Sine and Cosine transforms to obtain Fourier transforms.

Section 2 provides a traditional arm-waving proof of the Fourier Transform, while a more substantial proof using bare-bones distribution theory is presented in Appendix A. The Fourier Transform is discussed as a particular instance of Sturm-Liouville theory and the important claim is made that the exponential functions form a complete orthogonal basis on the interval $(-\infty,\infty)$.

Section 3 derives the Convolution Theorem,

$$a(t) = \int_{-\infty}^{\infty} dt'\, b(t-t')c(t') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega)\, C(\omega) \qquad (3.6)$$

with comments about dimensions of the various functions, integral operators, diagonalization, and the generalization to Fourier analysis on continuous groups.

Section 4 essentially applies the Convolution Theorem to obtain the theory of filters and their transfer functions in the ω domain. The connection is made between the time-domain differential equation describing a filter and the Green's Function or propagator which is seen to be the response of the filter to a time domain δ impulse. A simple RC filter is treated in detail, then a second example is the same filter in the limit that RC is very large.

Section 5 discusses the set of convention choices we made in stating the Fourier Integral Transform, and why those choices were made: sign of the exponential phase, i versus j, and allocation of the 2π.

Section 6 presents a version of the Fourier Transform and Inversion which we loosely refer to as "the generalized Fourier Transform". In the inversion formula, the ω contour is displaced in such a way as to increase the size of the class of causal functions for which Fourier Transforms exist. The main feature here is the analytic continuation of a spectrum X(ω) off the real axis in the complex ω plane. When the variable substitution s ≡ iω is made, the generalized Fourier Transform applied to causal functions becomes the traditional Laplace Transform,

$$\mathcal{X}(s) \equiv \mathcal{L}[x(t), s] \equiv X(s/i) \qquad (6.8)$$

$$\mathcal{X}(s) = \int_0^\infty dt\, x(t)\, e^{-st} \qquad (6.9)$$

$$x(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} ds\, \mathcal{X}(s)\, e^{+is} \qquad (6.10)$$

Section 7 states various reflection rules involving the Fourier transform and defines a Hermitian function.

Section 8 presents several simple examples of Fourier transforms which show how violation of the Fourier requirements causes transforms to be distributions instead of functions. The last example computes the spectrum of the Heaviside step function θ(t) (a distribution) and briefly introduces the notions of principal part integration and spectra presented in the form of limits. Details appear in Appendix C.

Section 9 considers a simple box-shaped x(t) and its Fourier transform X(ω). This particular example plays a major role in many later Sections.

Section 10 derives the various Parseval's Formulas and shows how one of these formulas relates to the total energy in a pulse.

Section 11 derives the differentiation rule dx(t)/dt ↔ [iωX(ω)] and presents a few examples.

Section 12 derives the time translation rule $x(t - t_1) \leftrightarrow X(\omega)\, e^{-i\omega t}$.

Section 13 derives these two "exponential sum rules",

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \qquad -\infty < k < \infty \qquad (13.3)$$

which are explored in more detail in Appendix A. These rules are important tools used later in processing expressions involving the power spectra of pulse trains.

Chapter 2: Pulse Trains and the Fourier Series Connection (17 p)

Chapter 2 examines the Fourier Transform spectrum of a simple pulse train formed from a general pulse shape. Consideration of pulse trains of infinite length leads to a derivation of the Fourier Series Transform. The chapter concludes with a discussion of sample pulse trains formed from box and bi-phase pulses.

Section 14 computes the Fourier Transform spectrum of a simple pulse train ($y_n = 1$).

Subsection (a) does this for an infinite pulse train. Although the spectrum of $x_{pulse}(t)$ is continuous, that of the pulse train is entirely discrete and consists of a set of delta function spectral lines.

Subsection (b) repeats the calculation for a finite simple pulse train, and this time the spectrum is found to be continuous. One can see exactly how this spectrum approaches the delta function lines as the length of the pulse train becomes infinite.

Section 15 in effect derives the traditional Fourier Series Transform from the Fourier Integral Transform of Chapter 1.

Section 16 obtains the spectrum of an infinite simple pulse train constructed from identical box pulses of width τ with rising edges separated by T_1. As noted above, the spectrum is entirely discrete.

Section 17 develops some graphical aids to help understand pulse train spectra. The discrete delta function spikes track a certain envelope function which is in fact the pulse spectrum $|X_{pulse}(\omega)|$. In certain simple cases, some of the delta spikes are missing because these spikes align with zeros of the pulse spectrum.

Section 18 discusses superposing a negative DC offset onto a positive box pulse train.

Section 19 constructs pulse trains using a certain "biphase pulse" instead of the box pulse. The spectra of these two pulse train types are compared. In a certain limit, the two become the same.

Chapter 3: Sampled Signals and Digital Transforms (50 p)

Chapter 3 deals with various digital forms of the Fourier Transform and their corresponding convolution theorems and applies these concepts to amplitude-modulated pulse trains (PAM signals) and to digital filters. Topics include image spectra, aliasing and Nyquist rate, group delay, FIR and IIR filters, poles, and impulse response. The first digital transform is called the Digital Fourier Transform which is an ω-domain version of the Z Transform whose variable is $z = e^{i\omega\Delta t}$. The Z Transform and Discrete Fourier Transforms are then addressed for both periodic and aperiodic signals. A recurring example is a simple RC filter section.

Section 20 considers an amplitude-modulated pulse train $x(t) = \Sigma_n\, y(t)\, T_1\delta(t - nT_1)$. The spectrum of the pulse train is found to be $X(\omega) = \Sigma_m\, Y(\omega - m\omega_1)$, and image spectra arrive on the scene. The notions of aliasing and the Nyquist rate are described with some traditional drawings.

Section 21 addresses two filter topics: digital filter image spectra, and analog filter group delay.

Subsection (a) considers a digital filter as an approximation to a standard "LIT" analog filter. What happens to the convolution theorem in this approximation? The answer is given in (21.13) and we find that such a digital filter has image spectra. The notion of aliasing is illustrated in Fig 21.1.

Subsection (b) shows that, when a narrow pulse passes through an analog bandpass filter, it experiences group delay $\tau_d = d\varphi/d\omega$ where φ is the filter phase. It then follows that filters with linear phase have a constant group delay, which is desirable to maintain the fidelity of pulses passing through the filter. It is claimed that the same notion applies to digital filters.

Section 22 is the first of two sections devoted to what we call the Digital Fourier Transform $X'(\omega)$. This transform is stated above (22.6) and is compared side-by-side with the Fourier Integral Transform $X(\omega)$. The time domain *digital* convolution sum $a = b * c$ of (22.6) (a digital filter) is found to have the simple diagonalized appearance $A'(\omega) = B'(\omega)C'(\omega)$ in the ω domain.

Section 23 concludes the presentation of the Digital Fourier Transform X'(ω).

Subsection (a) shows that the relation between X'(ω) and X(ω) is $X'(\omega) = \Sigma_m X(\omega - m\omega_1)$, which is exactly the form of main spectrum plus image spectra which occurred in both Sections 20 and 21. We then realize that the pulse train spectrum of Section 20 is just X(ω) = Y'(ω), where Y'(ω) is the Digital Fourier Transform of the signal y(t) whose samples y_n are the amplitudes of an amplitude-modulated pulse train whose pulses are delta functions. Similarly, we realize that A(ω) = B'(ω)C(ω) is a compact way to write the frequency domain digital filter equation where B'(ω) has image spectra.

Subsection (b) contains a "summary box" for the Digital Fourier Transform.

Section 24 shows that the Digital Fourier Transform is really the Z Transform in disguise. Specifically, the Z Transform X"(z) = (1/Δt) X'(ω) where $z \equiv e^{i\omega\Delta t}$. This last equation defines an analytic mapping between the ω plane and the z plane, as shown in Fig 24.1, which removes the redundancy present in the ω-plane picture. Various aspects of the Z Transform are then considered:

Subsection (a) states the digital convolution theorem in terms of the Z Transform.

Subsection (b) discusses the digital "unit impulse" signal in Z Transform notation.

Subsection (c) expresses time translation in Z Transform language, and here appears the famous notion that when a time domain signal is delayed one sample period, its Z Transform is multiplied by z^{-1}.

Subsection (d) addresses the notion of a digital time derivative and its Z Transform.

Subsection (e) uses this derivative idea to analyze a *digital* RC filter. The impulse response of such a filter is compared with that of the analog RC filter studied back in Section 4. The responses are shown to be the same in the limit Δt→0.

Subsection (f) considers a general polynomial ratio form for a digital filter transfer function H"(z). It is shown that, in a certain class of cases, the absence of (non-zero) poles in this ratio results in a filter whose impulse response decays in a finite amount of digital time. Conversely, the presence of such poles results in a never-ending impulse response. These are the FIR and IIR filters and it is shown exactly how these are represented in z space and the time domain, where these filters are implemented with registers, constant multipliers and adders. When H"(z) has poles, the filter is IIR and has feedback. When H"(z) has no poles, the filter is FIR and has no feedback. [FIR/IIR mean Finite/Infinite Impulse Response]

Subsection (g) revisits the digital RC filter considered in (e), and draws the z domain circuit in the standard IIR form Fig 24.7. An improved version of the digital RC filter is then trivially obtained.

Subsection (h) very briefly connects the idea of FIR and IIR filters with polynomial multipliers and dividers. Such circuits appear in scramblers and cyclic code error detection and correction algorithms.

Subsection (i) presents a "summary box" for the Z Transform

Section 25 considers the amplitude-modulated pulse train $x(t) = \Sigma_n y_n x_{pulse}(t - t_n)$ with an arbitrary pulse shape. It is shown that the Fourier Transform spectrum of such a pulse train is $X(\omega) = X_{pulse}(\omega)\, Y''(z)$ where $X_{pulse}(\omega)$ is the Fourier Transform spectrum of the pulse, while Y"(z) is the Z Transform of the sequence of pulse amplitudes y_n, where Y"(z) = (1/Δt) Y'(ω). A summary box appears as equation (25.4).

Example 1 is a specific pulse train having a short sequence of amplitudes y_n with a box-shaped pulse. The Digital Fourier Transform |Y'(ω)| is plotted using Maple, then $|X_{pulse}(\omega)|$ and |X(ω)| are plotted as well. The amplitudes y_n are explicitly recovered from the Digital Fourier Transform inversion formula, and the entire pulse train x(t) is explicitly recovered from the computed X(ω) using (1.2). This Example shows that the Digital Fourier Transform is more appropriate for making frequency domain plots than the Z Transform and really does have a role to play in digital signal analysis.

Example 2 considers a pulse train having just a single box pulse of amplitude y_0 at t = 0. The formula $Y'(\omega) = \Sigma_m Y(\omega- m\omega_1)$ is examined in light of the fact that $Y'(\omega) = y_0T_1$ = a constant. We find an unusual sum rule for the sinc function which states that $\Sigma_m \text{sinc}[\pi(x\text{-}m)] = 1$ for any real value of x, and this result is verified using an obscure summation formula found in Gradshteyn and Ryzhik.

Section 26 considers a y_n amplitude-modulated pulse train whose box-shaped pulses fill only a portion τ of the sample time T_1 , the so called "aperture" of a D/A converter. It is shown that for any finite aperture size the D/A converter output pulse train spectrum $X(\omega)$ differs from the desired signal spectrum $Y(\omega)$ not just in terms of image spectra (which can be filtered away), but also due to a sinc function distortion of the spectrum: $X(\omega) \sim \text{sinc}(\omega\tau/2)Y(\omega)$. This sinc distortion must be compensated by implementation of a "sine x over x" filter somewhere in the system, sometimes in an analog post-filter.

Section 27 gives a derivation of yet another transform, the Discrete Fourier Transform or DTF. The new feature here is that the pulse $x_{pulse}(t)$ used to create a pulse train is approximated by a sequence of N digital samples. Up to this point, our pulse trains have always been constructed from analog $x_{pulse}(t)$ functions having discrete amplitudes y_n.

Subsection (a) first reviews the Fourier Series Transform for an infinite simple pulse train built from analog pulses, $x(t) = \Sigma_n x_{pulse}(t\text{-}mT_1)$. When this pulse train is considered only at discrete times t_n where there are N values of t_n within each T_1 period, we end up with what we call the Discrete Fourier Transform of a Simple Pulse Train where the infinite set of Fourier Series c_m coefficients are replaced with a set of c'_m coefficients (the DFT coefficients) which have the property $c'_{m+N} = c'_m$, which means the sequence of c'_m coefficients is periodic and contains only N distinct values. The two halves of this pulse train transform are given in equations (27.9) and (27.11),

$$c'_m \equiv (1/N) \sum_{n=-\infty}^{\infty} x_{pulse}(t_n)\, e^{-imn(2\pi/N)} . \qquad \text{projection = transform} \qquad (27.9)$$

$$x(t_n) = \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} . \qquad \text{expansion = inverse transform} \qquad (27.11)$$

Subsection (b) is a proof that the above DFT transform is correct. In this brute-force proof, a rather strange identity is required which is obtained in Appendix B. A summary box for the Discrete Fourier Transform of a Simple Pulse Train is then given as (27.16).

Subsection (c) shows how the results of subsection (a) may be specialized to describe the Discrete Fourier Transform is a signal which is a single digitized pulse having N samples x_n and time duration T_1. This results in the traditional form of the Discrete Fourier Transform as summarized in box (27.21). Allowing for an arbitrary convention factor A, this DFT can be stated as

$$c'_m \equiv (A/N) \sum_{n=0}^{N-1} x_n\, e^{-imn(2\pi/N)} \qquad m = 0,1...N\text{-}1 \qquad \text{projection = transform}$$

$$x_n = (1/A) \sum_{m=0}^{N-1} c'_m\, e^{+imn(2\pi/N)} \qquad n = 0,1,...N\text{-}1 \qquad \text{expansion = inverse transform} \qquad (27.22)$$

Commonly appearing values of A are 1, N and $\sqrt{N}$.

Subsection (d) provides a graphical view of the DFT situation for a pulse train. One can consider the N pulse sample values as a vector $\mathbf{x_{pulse}}$ and the N coefficients as another vector **c'**, and these vectors are then related by an NxN symmetric matrix -- just another way to think about the above transform.

Chapter 4: Some Practical Topics (18 p)

Chapter 4 shows that symmetric FIR filters have linear phase and thus constant group delay. A specific brick wall digital filter is designed and then later used as an oversampling interpolation filter in the design of a D/A converter output section. This system is then simulated with a simple Maple program.

Section 28 shows that FIR filters have linear phase and thus constant group delay if the filter coefficients are symmetric. The result is demonstrated by two different methods.

Section 29 describes a specific design for a digital brick-wall filter with a 4x clock rate. A hardware implementation is shown in some detail, and the filter spectrum is computed and plotted. It is shown how 101 taps gives a better brick wall than 21 taps, though both exhibit the Gibbs phenomenon.

Section 30 considers a set of increasingly complicated D/A converter output section designs.

Subsection (a) describes a very simple D/A converter output section. A certain specific digital signal y_n is assumed in Fig 30.1. Both the Digital Fourier Transform spectrum |Y'(ω)| and Fourier Transform spectrum |X(ω)| of the stepwise analog output are computed and plotted. The first plot shows the expected image spectra, while the second shows a large sinc distortion of the output with the image spectra damped down significantly.

Subsection (b) clocks the D/A converter of the previous design at a 4x rate. This has no effect on the analog output waveform, but allows for a reinterpretation of the output in terms of a 4x rate analysis. This section is mainly an exercise in computing the output spectrum two different ways.

Subsection (c) adds a zero-stuffing input section to the converter design, which results in a new sampled signal with a 25% aperture as shown in Fig 30.8. Both |Y'(ω)| and |X(ω)| are computed for this new signal. The sinc distortion is much reduced, but several image spectra appear in the output spectrum.

Subsection (d) adds the 4x-rate brick-wall filter of Section 29 between the zero-stuffing input circuit and the D/A converter output circuit. The new X(ω) spectrum of the output shows the desirable features of a reduced sinc distortion and image spectra rejection. The system is then simulated with simple Maple code and the time-domain output plotted first for a 21-tap filter and then for a 41-tap filter. It is seen why this kind of filter is called an interpolation filter and an alias-rejection filter.

Chapter 5: Some Theoretical Topics (10 p)

Chapter 5 contains only **Section 31** which explores the subject of dispersion relations for the spectral function X(ω) and for other related functions treated as analytic functions of a complex variable.

Subsection (a) derives an integral equation (32.1) which X(ω) must satisfy if X(ω) is analytic in the upper half ω plane. The integral in this equation is a "principle value" integral which has a tick mark. This type of integral is described more in Appendix C.

Subsection (b) derives a similar equation (31.3) for X(ω) being analytic in the lower half ω plane. These two equations are then written in (31.5) as one equation with a parameter $\sigma = \pm 1$ to distinguish the cases. The RC filter spectrum G(ω) found in Section 4 (b) is shown to satisfy this integral equation.

Subsection (c) rewrites the dispersion relation in terms of the real and imaginary parts of X(ω), shown in (31.8), so now there are two equations. In the case that X(ω) is associated with a real signal x(t) the negative frequency half of the integral can be folded into the positive half to give a new form (31.9) which is associated with the names Kramers and Kronig.

Subsection (d) shows how the dispersion relations are also valid for γ(ω) where $X(\omega) = e^{-\gamma(\omega)}$, with certain assumptions. The real and imaginary parts of $\gamma(\omega) = \alpha(\omega) + i\,\beta(\omega)$ are the attenuation and phase of the spectrum (or transfer function) X(ω). When those assumptions are met, such a filter is a minimal phase filter. It is shown that the dispersion relations mix together the attenuation and phase, so neither can be set independently of the other.

Subsection (e) demonstrates that a minimal phase filter has approximate linear phase in any region of ω that is far away from regions where the attenuation α(ω) significantly varies.

Subsection (f) applies this idea to a zero resistance coaxial cable and argues that such a cable has linear phase up to infrared frequencies. Linear phase means constant group delay which means the cable is non-dispersive. Real cables of course don't have zero resistance and exhibit skin effect.

Subsection (g) rewrites the dispersion relations in terms of the Hilbert Transform. The dispersion relation becomes roughly the statement that X(ω) must be ±i times its own Hilbert Transform. It is emphasized that the dispersion relation is only a condition on X(ω) and does not fully determine X(ω).

Chapter 6: Power in Pulse Trains (90 p)

Chapter 6 derives expressions for the energy and power, and the spectral energy and power densities of an amplitude-modulated pulse train. This work is carried out with a moderate amount of mathematical rigor. Correlation and autocorrelation are mentioned. The notion of a statistical pulse train is presented and the spectral power density of such pulse trains is established. These results are then applied to various uncorrelated standard line codes including NRZ, RZ and Manchester. Two examples of correlated pulse trains are then treated -- AMI and Change/Hold -- and the latter is then used to get results for the NRZI line code.

Section 32 introduces the autocorrelation function $r_x(t)$ of function x(t) and discusses energy and power.

Subsection (a) computes $r_x(t)$ for a square pulse.

Subsection (b) defines quantities E, p(t) and $\mathcal{E}(\omega)$ which are the total energy, instantaneous power, and spectral energy density of a pulse train.

Subsection (c) shows that the Fourier Integral Transform $R_x(\omega)$ of $r_x(t)$ is directly related to $\mathcal{E}(\omega)$, a fact known as the Wiener-Khintchine relation.

Subsection (d) verifies this theorem for the square pulse.

Subsection (e) digresses on the subject of cross-correlation and relates this to auto-correlation.

Section 33 computes the power spectral density of a *simple* pulse train with arbitrary $x_{pulse}(t)$.

Subsection (a) computes $|X(\omega)|^2$ for an infinite simple pulse train.

Subsection (b) repeats this calculation for a finite simple pulse train.

Subsection (c) defines the spectral power density $\mathscr{P}(\omega)$ of a pulse train and displays the result for the infinite and finite pulse trains based on the $|X(\omega)|^2$ computations of the previous two subsections. As expected, the infinite pulse train has a discrete power spectrum while the finite one has a continuous spectrum. In the infinite case, $\mathscr{P}(\omega)$ is related to the Fourier Series coefficients c_m , a_m and b_m.

Subsection (d) defines the average power P of a pulse train and relates that to c_m , a_m and b_m .

Section 34 computes the spectral power density of a *general* (PAM) pulse train with arbitrary $x_{pulse}(t)$.

Subsection (a) gathers up in a summary box a set of energy and power facts already derived which apply to general pulse trains, not just simple ones.

Subsection (b) computes the spectral energy density $\mathcal{E}(\omega)$ and power density $\mathscr{P}(\omega)$ for a general pulse train. The results are stated for infinite pulse trains, with the understanding that all sums are replaced by finite sums for finite pulse trains.

Subsection (c) uses two methods to compute the spectrum $X(\omega)$ and power spectral density $\mathscr{P}(\omega)$ of a general pulse train in which the amplitudes have the form A,B,A,B.... with arbitrary $x_{pulse}(t)$. The first method involves a brute force calculation of the result for a finite pulse train, and then takes the limit $N\rightarrow\infty$ to obtain the result for an infinite pulse train. The second method treats the pair of pulses A,B as a single pulse of period $2T_1$ and obtains the same results using a Fourier Series analysis. The results are then applied to a short catalog of standard cases (such as A = 1, B = 0) and displayed in Figures 34.2 through 34.9. At the very end results are stated for repeat sequences A,B,C and $A_0,A_1....A_{M-1}$. The general case is treated in Appendix F.

Section 35 treats statistical pulse trains in which the amplitudes take "random" values. For example, a square wave pulse train might have probability p for amplitude 1 and 1-p for amplitude 0.

Subsection (a) defines the notion of a statistical pulse train in terms of an ensemble of pulse trains which have random variables Y_n associated with each position in the pulse train. The horizontal and vertical averages $\langle\rangle_1$ and $\langle\rangle$ are defined with a simple example.

Subsection (b) shows the region in (m,s) space for which $y_m y_{m+s}$ is non-zero for a finite pulse train, and states the corresponding restriction on the autocorrelation sequence r_s.

Subsection (c) expresses the spectral power density $\mathscr{P}(\omega)$ in a variety of ways first for infinite pulse trains and then for finite ones. These two groups of equations are repeated below with modifications.

Subsection (d) takes the ensemble average of the equation groups of subsection (c).

Subsection (e) shows how the two equation groups simplify if stationarity is assumed.

Subsection (f) shows how the two equation groups further simplify if stationarity and statistical independence is assumed.

Subsection (g) comments on distinctions between the $\langle\rangle_1$ and $\langle\rangle$ averages. It then shows that for a large ensemble of infinite pulse trains one has $\langle\rangle_1 = \langle\rangle$, so the distinction between the two kinds of averages goes away. They are approximately equal for very long pulse trains.

Subsection (h) first discusses two general methods for computing the spectral power density $\mathscr{P}(\omega)$. These are the autocorrelation method and the double-sum method. It is then noted that for a very large ensemble of very long pulse trains, $\langle\mathscr{P}(\omega)\rangle = \mathscr{P}(\omega)$. Finally, various Facts about an infinite uncorrelated pulse train (stationary and independent) are collected. The Facts are then applied to pulse trains whose amplitudes take values in the set {A,B}.

Subsection (i) displays a summary box for finite and infinite uncorrelated statistical pulse trains and then provides a simple example.

Subsection (j) presents a numerical simulation of an ensemble of random pulse trains. A simple Maple program computes and plots the average spectral power density $<\mathcal{P}(\omega)>$ of the ensemble, and one clearly sees the continuous and discrete pieces of the spectrum. $<X(\omega)>$ is also computed.

Subsection (k) considers and then resolves an interesting paradox: the simulation shows a power spectrum $<\mathcal{P}(\omega)>$ having a continuous component, whereas the simulated $<X(\omega)>$ has no continuous component. This seems strange since $\mathcal{P}(\omega) \sim |X(\omega)|^2$.

Subsection (ℓ) discusses the role which the autocorrelation function/sequence has played in this document's presentations.

Section 36 computes the spectral power density for several standard uncorrelated line codes. This includes unipolar and bipolar versions of the NRZ and RZ lines codes, and the Manchester line code. For each line code the spectrum is stated and plotted, and the way power partitions into AC (lines and continuous) and DC components is reviewed. Eye patterns and ISI receive a passing comment.

Section 37 computes the spectral power density for the Alternate Mark Inversion (AMI) line code. In this line code, the locations in the pulse train are correlated with each other which makes the calculation of $<y_m y_n>$ more difficult -- one cannot claim $<y_m y_n> = <y_m><y_n>$. The autocorrelation sequence is calculated, plotted, and is then used to compute the spectral power density $\mathcal{P}(\omega)$. Results are summarized in the last subsection, the spectrum is plotted, and various limits of the spectrum are explored.

Section 38 repeats the previous section for what we call the Change/Hold line code. Again there is correlation among the amplitudes y_n. The autocorrelation sequence is calculated, plotted, and is then used to compute the spectral power density $\mathcal{P}(\omega)$. Results are summarized in the last subsection, the spectrum is plotted, and various limits of the spectrum are explored.

Appendix A: Delta Function Technology (19 p)

Appendix A discusses "delta functions" at a someone deeper and more practical level than one commonly finds in texts and on the web. Delta function models are constructed and many mathematical identities are developed which find use in the main text. It seemed best to isolate this material into an appendix rather than derive its results in line.

The opening text gives a very minimalist outline of "distribution theory", with a mention of the meaning of "symbolic functions" like the delta function.

Subsection (a) develops four different "models" (δ_1 through δ_4) for the delta function. Each model is a sequence of functions which approaches the true δ in the limit of some parameter. As part of this development, two different proofs are given for equation (2.1),

$$\int_{-\infty}^{\infty} dx\, e^{\pm ikx} = 2\pi\delta(k) \quad . \qquad (2.1)$$

Subsection (b) develops models δ_5, δ_6 and δ_7 for what we call "periodic delta functions".

Subsection (c) derives the following two "exponential sum rules"

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \qquad (13.2)$$

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \qquad -\infty < k < \infty \qquad (13.3)$$

and it is shown how the first is a limit of the second. Equation (13.2) is then used to derive the Poisson Sum Formula. Both sum rules are used many times in the main text.

Subsection (d) notes that the number $\delta(0)$ is undefined in distribution theory, but is meaningful for a specific delta function model. It is just a notation to allow us to handle certain infinite sums that occur many times in the main text. A related notion is called "undoing the limit", or "backing off". For example, one can undo the limit of (13.2) above to arrive at (13.3) in which the right side is a perfectly normal function. Sometimes it is necessary to undo the limit in this way, do some calculations, and then take the limit $N\rightarrow\infty$ again.

Subsection (e) considers the delta function sifting property (2.2) and pays particular attention to what happens at the two integration endpoints. A special notation $\Theta(a \le x \le b)$ is introduced to incorporate the endpoint effects. Several properties of the Θ "function" are stated.

Subsection (f) addresses the tricky subject of the product of two delta functions of the same variable. This concept is undefined in distribution theory, but one must face such products when computing objects like the spectral power density of a pulse train: the spectrum includes delta functions, and the power density is proportional to the spectrum squared. The solution as mentioned just above is to back off from the $N=\infty$ limit, figure out what is happening, and then resume the limit again. It is in this process that the delta function models δ_5 and δ_6 find their use.

Appendix B: Derivation of a Certain Identity (2 p)

Appendix B derives this rather obscure identity, where N and s are arbitrary integers with $N > 0$,

$$\sum_{m=0}^{N-1} e^{+ims(2\pi/N)} = N \sum_{m=-\infty}^{\infty} \delta_{s,mN} \qquad \textit{(B.1)}$$

Appendix C: The Fourier Transform and its relation to the Hilbert Transform (17 p)

Appendix C further develops Fourier Integral Transform theory beyond the treatment of the main text. Instead of using the notation $f(t)$ and $F(\omega)$, here we use $f(t)$ and $f\hat{}(\omega)$.

Subsection (a) introduces this ^ notation for the Fourier transform and allows for a general constant k in the transform definition to allow comparison of results with sources using other conventions. In this new notation, some "Facts" are derived, such as

$$[f(-u)]\hat{}(-\omega) = f\hat{}(\omega) \qquad f\hat{}^{-1}(u) = (1/2\pi k^2)\, f\hat{}(-u) \qquad f\hat{}\hat{}(t) = 2\pi k^2 f(-t) \ .$$

An operator notation is introduced which is similar to that commonly used for the Laplace Transform.

Subsection (b) introduces the notion of a Cauchy Principle Value integral with a preliminary look at its associated symbolic function pf(x). We use the tick notation $⨍$ for such an integral.

Subsection (c) computes the "regular" Fourier transform of the function f(u) = 1/u, which is not quite as simple as it seems. It is shown that (1/u)^(ω) = -iπk sgn(ω) where sgn(ω) = 2θ(ω) – 1 and where θ(ω) is the Heaviside step function. The inverse Fourier transform is then done to verify that the original function 1/u is recovered. A directly related result is [sgn(ω)]^(t) = -2ik(1/t).

Subsection (d) gives a derivation of this fact involving complex integration (ω is real) ,

$$\lim_{\varepsilon\to 0} \frac{1}{\omega \mp i\varepsilon} = \mathrm{pf}(1/\omega) \pm i\pi\delta(\omega) \qquad \text{(C.22)}$$

which for want of a better name we call The Pole Avoidance Rule. A certain contour avoids hitting a pole by deflecting one way or the other around the pole. Along the way, pf(x) is defined as a symbolic function.

Subsection (e) computes the Fourier transform of the function $\lim_{\varepsilon\to 0}$ [1/(u±iε)] which seems very close to the function 1/u explored in subsection (c). One finds that $(\frac{1}{u\pm i\varepsilon})\hat{}(\omega) = \mp 2\pi ik\ \theta(\pm\omega)$. The original function is then recovered using the inverse Fourier transform. A directly related result is the fact that $[\theta(\pm t)]\hat{}(\omega) = k\ \mathrm{pf}(\frac{1}{\pm i\omega}) + \pi\ k\ \delta(\omega)$ (Fourier transform of the Heaviside step function).

Subsection (f) then computes [θ(u)]^(ω) using the "generalized" Fourier transform (in which the recovery contour is vertically shifted). The result is $[\theta(u)]\hat{}(\omega) = \frac{k}{i\omega}$.

Subsection (g) summarizes all the examples considered in this appendix.

Subsection (h) defines the Hilbert Transform $f^h(t)$ of a function f(t). This transform has an intimate connection with the Fourier Transform which is laid out in a series of unusual facts such as

$$[f^h]\hat{}(\omega) = k^{-1}\ (\frac{1}{\pi t})\hat{}(\omega)\ f\hat{}(\omega) = -i\ \mathrm{sgn}(\omega)\ f\hat{}(\omega) \qquad\qquad f^{hh}(t) = -f(t)\ .$$

The inversion formula for the Hilbert transform is derived and the transform is then computed for some simple examples. The Hilbert Transform appears in the dispersion relations of Section 31 (g).

Appendix D: Calculation of a Sum which appears in (35.17) (5 p)

This identity is derived,

$$\sum_{s=-2N}^{2N} \frac{2N+1-|s|}{2N+1}\ e^{\pm iks} = \frac{1}{2N+1}\ \frac{\sin^2[(N+1/2)k]}{\sin^2(k/2)} \qquad // = 2\pi\delta_6(k,N) \text{ of (A.20)} \qquad \textit{(D.1)}$$

The sum appears in the power spectrum of a finite pulse train when that spectrum is computed by the autocorrelation method.

Appendix E: Table of Transforms (4 p)

Lists all transform pairs which appear in this document and shows how they are related.

Appendix F: The Spectrum and Power Density for Repeated-Sequence Pulse Trains (20 p)

For infinite pulse trains composed of repeats of some length-P subsequence:
Subsection (a) computes the spectrum X(ω) in (F.12)
Subsection (b) computes the spectral power density $\mathscr{P}(\omega)$ in (F.23) in terms of $|Y_P"(z)|^2$.
Subsection (c) computes $<\mathscr{P}(\omega)>$ for an ensemble of pulse trains which respect the special condition

$$<y_m^* y_n> = \alpha \quad \text{for } m \neq n$$
$$<y_m^* y_n> = \beta \quad \text{for } m = n \qquad (F.25)$$

The result for $<\mathscr{P}(\omega)>$ is stated in (F.33) in several different forms.
Subsection (d) takes the P→∞ limit of the subsection (c) result for $<\mathscr{P}(\omega)>$.
Subsection (e) computes $\mathscr{P}(\omega)$ for a single pulse train which respects the special condition

$$<y_m y_n>_1 = \alpha \quad \text{for } m \neq n + NP \qquad N = \text{any integer}$$
$$<y_m y_n>_1 = \beta \quad \text{for } m = n + NP \qquad (F.43)$$

where $<y_m y_n>_1$ is a horizontal average across the single sequence (autocorrelation). The result for $\mathscr{P}(\omega)$ is stated in (F.52). It is noted that the results for $<\mathscr{P}(\omega)>$ of subsection (c) and $\mathscr{P}(\omega)$ of subsection (e) are exactly the same in terms of their respectively defined α and β constants and the reason is given.

Subsection (f) summarizes the power spectrum and its conditions as Fact (F.54). A graphical representation is drawn for the spectrum in general, and for a box pulse in particular. It is shown that the MLS sequence is a candidate for application of this Fact, and the MLS spectrum is stated.

Subsection (g) treats the P = 2 repeated subsequence A,B using the general formulas of subsections (a) and (b)

Appendix G: Random Variables, Probability Theory and Pulse Train Amplitudes (24 p)

Subsection (a) provides an opening definition of a "random variable" as being a parameter associated with a probability distribution. The terms experiment, outcome, sample space and event are defined and two simple classic experiments considered: rolling a die, and spinning a spinner. The corresponding pdf and pmf distributions are discussed.

Subsection (b) refines the definition of "random variable" as a mapping from the sample space to the real numbers, and the coin toss experiment is used as an example. Drawings show examples of discrete and continuous sample spaces and the random variable mappings.

Subsection (c) contains a brief review of basic probability theory with an emphasis on notation. The notions of statistical independence, correlation and normalization are discussed. Some Math Lemmas are presented relating the ordering of terms in a sum $\Sigma_{x,y}$ in the context of some function z = f(x,y) by first summing over a surface of constant z, then over different z values, so $\Sigma_{x,y} = \Sigma_z \Sigma_{surface(z)}$, These lemmas are needed in the following discussion of random variables defined as functions of other random variables. After this, the usual suspects of probability theory are rolled out for inspection: expected values, mean, covariance, variance, standard deviation and correlation.

Subsection (d) treats "ensemble experiments" such as rolling N loaded dice, and a transition is then made to the analysis of random variables associated with an ensemble of people.

Subsection (e) treats the rolling of two weighted, magnetically interacting dice, after which special cases are considered.

Subsection (f) gives examples of the computation of probability mass density functions from experimental data.

Subsection (g) applies the notions of random variables and probability theory to the sequences which are the amplitudes of pulse trains.

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